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HADRONS

review work

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Contents

Chapter 1

Introduction

1.1 Abstract

This project compiles in general the present status of knowledge about this group of subatomic particles. It also shows important milestones in the process of their recognition. The history of hadrons discovery will be recapitulated here. Theories attempting to classify them are presented. The last evolutional step is the quark model. Mathematical framework of groups and algebras necessary for the description of theories is exposed in the last chapter. It will be referred to it's conclusions in the text. Every particle has its own history of discovery. I believe it's discoverer is a part of it and if we want to understand the process of discovery, we have to familiarize ourselves with persons involved. So, short biographies of the leading personalities are included.

1.2 What are hadrons?

The name "hadron" comes from the Greek word for "strong"[6]. These are particles built from quarks and experiencing the strong nuclear force. Except proton, all other hadrons (free from forces) are unstable. Even if proton was not absolutely stable, experiments show, that it's lifetime would be in excess of 10^{32} years. Neutron is the second most stable hadron with lifetime about 16 minutes. Most hadrons, however, last for less than 10^{-8} seconds. If the lifetime of a hadron is more than 10^{-20} we denote it as "stable". It does not mean the same as for proton, it is used to differentiate ordinary hadrons from shorter-lived hadrons (which we merely do not consider as a particle, we rather call it resonance for it's short lifetime $\sim 10^{-23}s$). Stable hadrons usually decay via weak and electromagnetic force. The very short-lived hadrons decay via the strong force, but this force is so strong that it allows the particle to live only for about the time it takes light to cross the particle. We can observe them only as a resonance phenomenon during the decay(that is the reason for the name resonances).

Hadrons, whether stable or resonant, fall into two classes:baryons and mesons. Originally the names referred to the relative masses of the two groups of particles. The baryons (from the Greek word for "heavy") included the proton and heavier particles; the mesons (from the Greek word for "between") were particles with masses between that of electron and the proton. Now, however, the name baryon refers to any particle built from three quarks(with baryon number 1) and meson refers to any particle built from quark and antiquark(with baryon number 0). These are the only two combinations of quarks and antiquarks, that the strong binding force apparently allows. Figure 1.1[5] lists the stable baryons and mesons and indicates their spins, masses and mean lifetimes, as well as some of the principal ways in which they ultimately decay.

Table : Stable Hadrons

name	symbol	spin	mass (MeV)	mean life (s)	examples of possible decay modes
Mesons (Baryon number $B = 0$)					
Pion	$\pi^{\pm}$		140	3×10^{-8}	$\to \mu^{\pm}\nu_{\mu}$
	π^0		135	8×10^{-17}	$\to \gamma\gamma$
Kaon	$K^{\pm}$		494	1×10^{-8}	$\to \mu^{\pm}\nu_{\mu}$
	K^0		498	9×10^{-11}	$K_S \to 2\pi$*
				5×10^{-8}	$K_L \to 3\pi$*
D	$D^{\pm}$	0	1,869	1×10^{-12}	$\to K^{\pm} + \ldots$
	D^0		1,865	4×10^{-13}	$\to K^{-} + \ldots$
D_s	$D_s^{\pm}$		1,969	5×10^{-13}	$\to K^{\pm} + \ldots$
B	$B^{\pm}$		5,279	2×10^{-12}	$\to \mu^{\pm}\nu_{\mu} + \ldots$
	B^0		5,279	2×10^{-12}	$\to D^{-}\mu^{+}\nu_{\mu}$
Baryons (Baryon number $B = 1$)					
Proton	p		938	stable†	
Neutron	n		940	9×10^{2}	$\to pe^{-}\bar{\nu}_e$
Lambda	Λ		1,116	3×10^{-10}	$\to p\pi^{-}$
Sigma	Σ^{+}	1/2	1,189	8×10^{-11}	$\to p\pi^{0}$
	Σ^{0}		1,193	7×10^{-20}	$\to \Lambda\gamma$
	Σ^{-}		1,197	1×10^{-10}	$\to n\pi^{-}$
Xi, or cascade	Ξ^{0}		1,315	3×10^{-10}	$\to \Lambda\pi^{0}$
	Ξ^{-}		1,321	2×10^{-10}	$\to \Lambda\pi^{-}$
Omega-minus	Ω^{-}	3/2	1,672	8×10^{-11}	$\to \Lambda K^{-}$
Lambda-c	Λ_c^{+}	1/2	2,285	2×10^{-13}	$\to \Lambda + \ldots$

*The K^0 and its antiparticle $\bar{K}^0$ mix quantum mechanically to form two physical states, K_L and K_S. †Could be unstable according to grand unified theories, but experiments show that its lifetime is at least 10^{32} years.

Figure 1.1: Table of most important hadrons

Chapter 2

Discoveries of first hadrons

2.1 Ernest Rutherford

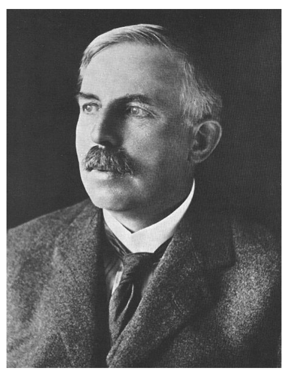

Figure 2.1: Lord Sir Ernest Rutherford, baron of Nelson, of Cambridge[19]

Lord Sir Ernest Rutherford, baron of Nelson, of Cambridge was born $30th$ of August 1871 at Brightwater near Nelson in New Zealand[16]. He studied at Nelson College in New Zealand, lately at Canterbury College in Christchurch, where he studied properties of iron in high-frequency alternating magnetic fields. In 1895 he came to Cavendish Laboratory at Cambridge University, where he began to work under J. J. Thompson on the detection of Hertzian waves. In December 1895, when Röntgen discovered X rays, Thomson asked Rutherford to join him in a study of the effects of passing a beam of X rays through the gas. In 1896 the French physicist Henry Becquerel discovered that uranium emitted rays which could fog a photographic plate as did X rays. Rutherford soon showed that they also ionized air but they were different from X rays, consisting of two distinct types of radiation. He named them alpha and beta rays. In 1898 Rutherford was appointed to the chair of physics at McGill

University in Montreal. The Royal Society awarded him the Rumford medal in 1904. In 1903 he showed that alpha rays could be deflected by electric and magnetic fields. Rutherford wrote 80 scientific papers during his seven years at McGill, made many public appearances, among them the Silliman Memorial Lectures at Yale University in 1905. In 1907 he returned to England to accept a chair at the University of Manchester, where he continued his research on the alpha particle. With Hans Geiger they counted the particles as they were emitted one by one from a known amount of radium. With his student Thomas D. Royds he proved in 1908 that the alpha particle really is a helium atom. Almost immediately, in 1908, came the Nobel Prize - but for chemistry, for his investigations concerning the disintegration of elements[19]. In 1911 Rutherford made his greatest contribution to science with his nuclear theory of the atom. A knighthood conferred in 1914 further marked the public recognition of Rutherford's services to science. He produced the first artificial disintegration of an element in 1919, when he found that through collisions with alpha particle the atom of nitrogen was converted into the atom of oxygen and the atom of hydrogen. In the second Bakerian lecture he gave to the Royal Society in 1920, he speculated upon the existence of the neutron and of isotopes of hydrogen and helium; three of them were eventually discovered by workers in the Cavendish Laboratory. In 1931 he was made a peer. Recalling is origins, he chose the title of "Baron of Nelson" and in the crest of his coat of arms he included a kiwi bird. The blazoning reads, "On a Wreath of the Colours upon a rock a Kiwi proper."[1] He died on 19*th* of October 1937 in Cambridge following a short illness and was buried in Westminster Abbey.

2.2 The Proton discovery process and properties

In Montreal Rutherford observed that fast-moving alpha particles on passing through thin plates of mica produced diffuse images on photographic plates, whereas a sharp image was produced when there was no obstruction to the passage of the rays. He considered that the particles must be deflected through small angles as they passed close to atoms of the mica. But calculation showed that an electric field of 10^8 volts per centimeter was necessary to deflect such particles traveling at $\frac{2}{3}c$, a most astonishing conclusion[1]. This phenomenon of scattering was found in the counting experiments with Geiger; Rutherford suggested to Geiger and a student, Ernest Marsden, that it would be of interest to examine whether any particles were scattered backward–i.e., deflected through an angle of more than 90 degrees. To their astonishment, a few particles in every 10000 were indeed so scattered. Rutherford came to the conclusion that the intense electric field required to cause such a large deflection could occur only if all the positive charge in the atom, and therefore almost all the mass, were concentrated on a very small central nucleus some 10000 times smaller in diameter than that of the entire atom. The positive charge on the nucleus

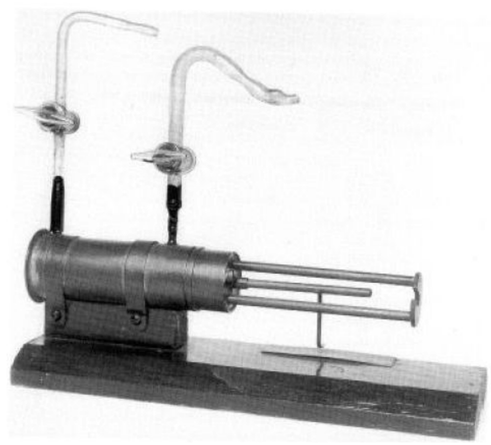

Figure 2.2: Rutherford's nuclear disintegration chamber[1]

would therefore be balanced by an equal charge on all the electrons distributed somehow around the nucleus. This theory of atomic structure is known as the Rutherford atomic model. It was not until 1913 that Niels Bohr, a Danish physicist, postulated that electrons do indeed move in orbits about a central nucleus, thus upholding the convictions of Rutherford[5]. After that, he adverted to the study of nucleus. For this, he had to break nucleus somehow and try to detect, what comes from it. Rutherford had noted earlier that a metal source coated with the alpha emitter radium C always gives rise to particles that produce scintillations on a zinc sulfide screen at a distance beyond the range of alpha particles in air. Studying this phenomenon in magnetic field, Rutherford concluded that the particles responsible for the scintillations were the nuclei of hydrogen. However, he did not know whether these nuclei were just recoiling nuclei from hydrogen atoms, that happened to be present on the metal source and were stuck by alpha particles, or whether they were actually knocked out of elements heavier than hydrogen. To study the phenomenon, he put a radium C source in an evacuated metal box with a hole covered by a very thin silver plate. The plate would allow the alpha particles to get out and strike a zinc sulfide screen, and yet would keep air out of box. Rutherford observed the change in the number of oscillations when various metal foils were placed between the silver plate and the zinc sulfide screen, or when various gases were admitted into the box. For the most part, the rate of scintillations decreased in proportion to the stopping power of the foils or gases. However, when dry air was admitted into the box, the scintillation rate went up. By repeating this experiment with all the constituents of air - oxygen, nitrogen,... - Rutherford learned (1917) that the effect was due to the collisions of alpha particles from radium C source with the nuclei of nitrogen in the air. The process Rutherford discovered was the disintegration of the nitrogen nucleus, in which an alpha particle penetrates into the nucleus and knocks out a ”hydrogen nucleus”. The reason that this had not been seen long before is that the electric repulsion between the positively charged alpha particles and a heavy nucleus like that of gold was just too strong to allow the alpha particle to get close to the nucleus. Nitrogen, on the other hand, has a nuclear charge of only seven units, so the exceptionally energetic alpha particles emitted by the radium C could at least get close to the nucleus, and could occasionally hit an outlying ”hydrogen nucleus”. In the report of this result in a 1919 paper, Rutherford concludes[1] the following:

”From the results so far obtained it is difficult to avoid the conclusion that the long-range atoms resulting from collisions of alpha particles with nitrogen atoms are not nitrogen atoms but probably atoms of hydrogen, or atoms of mass 2. If this is the case, we must conclude that the nitrogen atom is disintegrated under the intense forces developed in close collision with a swift alpha particle, and that the hydrogen atom, which is liberated formed a constituent part of the nitrogen nucleus. . . ”

In a famous talk in 1920, his second Bakerian lecture[16] before the Royal Society, Rutherford accepted the hydrogen nucleus as an elementary particle naming it proton, from Greek word ”protos,” which means ”the first”. He also speculated about new kinds of atomic nuclei, but he pictured them all as consisting of protons and electrons. One of the hypothetical nuclei about which Rutherford speculated was a ”neutron”(he named it so), with atomic weight 1 and electric charge 0, but this was still pictured as a composite of a proton and an electron. It was entirely unclear to anyone why some of the electrons in an atom should be bound in the nucleus while the others revolved in much larger orbits outside the nucleus, but no one had any idea anyway of what sort of force might be operating at an extremely short distances separating particles within a nucleus. Present status of proton properties says that the proton is stable subatomic particle with unitary positive charge $1,602.10^{-19}C$ and a mass of $1.67262.10^{-27}kg$ $(938,2720MeV)$, which is 1836 times the mass of electron. Magnetic moment is $\mu = 2,7928\mu_N$[7]. It belongs to the baryon family, that means it consists of three quarks uud and it has a baryon number 1. Antiprotons were first identified[7] in 1955 by Emilio Segré and Owen CHamberlain by bombarding a copper target with high energy protons at the University of California at Berkeley.

2.3 James CHadwick

Sir James Chadwick
was born in Cheshire, England, on $20th$ October, 1891[9]. He attended Manchester High School and Manchester University in 1908 and after graduation he spent the next two years under Professor Rutherford in the Physical Laboratory in Manchester, where he worked on various radioactivity problems. In 1913 he proceeded to Berlin under Professor H. Geiger. After the war, in 1919, he returned to England to Gonville and Caius College, Cambridge, and to resume

Figure 2.3: Sir James Chadwick[19]

work under Rutherford. In Cambridge, Chadwick joined Rutherford in accomplishing the transmutation of other light elements by bombardment with alpha particles, and in making studies of the properties and structure of atomic nuclei. In 1927 he was elected a Fellow of the Royal Society. In 1932, Chadwick made a fundamental discovery in the domain of nuclear science: he proved the existence of neutrons. For this epoch-making discovery he was awarded the Hughes Medal of the Royal Society in 1932, and subsequently the Nobel Prize for Physics in 1935[19]. He remained at Cambridge until 1935 when he was elected to the Lyon Jones Chair of Physics in the University of Liverpool. From 1943 to 1946 he worked in the United States as Head of the British Mission attached to the Manhattan Project for the development of the atomic bomb. He returned to England and, in 1948, retired from active physics. Sir James was knighted in 1945. He received several medals and he was a honorary member of about 15 universities and academies around the world. Sir James Chadwick died on July 24, 1974.

2.4 The Neutron discovery process and properties

For twenty years after discovery of the atomic nucleus, physicists generally thought that the nuclei of all elements consisted of hydrogen nuclei (later called proton) and electrons. Helium has atomic weight 4 and atomic number 2, so its nucleus was supposed to consist of four protons and two electrons, to give it a nuclear charge of $4 - 2 = 2$ electron units. To find out what the nucleus really consist of, it was necessary to break it up and see what came out. In 1930 two German physicists, Walther Bothe and Herbert Becker, had reported[9] that by bombarding beryllium nuclei with alpha particles from a polonium source they had registered the emission of a powerful neutral radiation, much more

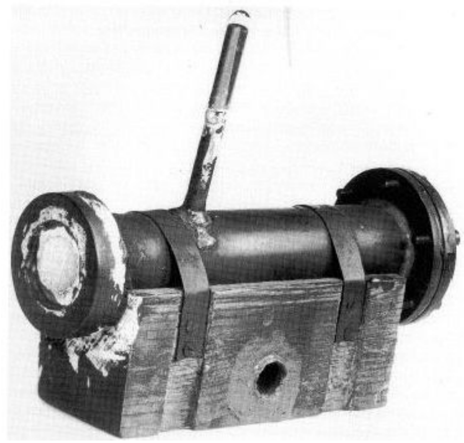

Figure 2.4: Chadwick's nuclear disintegration chamber[1]

penetrating than the protons emitted in nuclear disintegrations like those studied earlier by Rutherford. The rays were at first thought to be electromagnetic radiation. Later, Iréne and Frédéric Joliot-Curie reproduced the phenomenon and proved that the mysterious radiation could knock protons out of a hydrogen rich material, such as paraffin wax[1]. This might not have been surprising, but the protons were found to have a remarkably high speed. The Joliot-Curies calculated that if the rays emitted from beryllium were really electromagnetic radiation, the beryllium nucleus must be releasing ten times more energy than was carried by the alpha particle that produced the rays in the first place. The Joliot-Curies were even led to question whether the law of conservation of energy was being violated in these processes. When their "Note aux comptes rendus de l'Academie des Sciences", where they presented their results, arrived in Rome on $28th$ of January 1932, according to what Gian-Carlo Wick, who was present, said[9], Ettore Majorana exclaimed: "Stronzi(idiots), they have not understood that it is neutron!" Soon after Chadwick began to study the "beryllium rays", directing them into various other materials besides paraffin. He soon found that nuclei other than hydrogen would also recoil when stuck with these rays, but that they moved with a velocity much less than for hydrogen. The pattern of decreasing recoil velocities with increasing atomic weight of the recoiling nucleus was just what would be expected if the "beryllium ray" was not electromagnetic radiation but a particle with a mass close to that of proton. One other property of the "beryllium ray" particles was clear from the start: their great penetrating power meant that they must be electrically neutral. It seemed from its atomic weight and its neutrality, that the particle produced by alpha rays in beryllium was just the electrically neutral composite of a proton and an electron about which Rutherford had speculated in his Bakerian lecture[1] in 1920. Chadwick reported this result to the Kapitza Club, an informal circle of physicists that

had been brought together at the Cavendish laboratory by the Russian physicist Peter Leonidovich Kapitza. A few days later Chadwick published the discovery in Nature(February 27, 1932) and a little later in the Proceedings of the Royal Society, where he officially called it neutron.
For Chadwick as for Rutherford, the neutron was merely a composite of a proton and an electron, not an elementary particle in its own right. He also did not speculated about the role of the neutron in the structure of nucleus. This problem was taken up by Werner Heisenberg. In a series of 1932 papers Heisenberg proposed[1] that nuclei consist of protons and neutrons and are held together by the exchange of electron between them. That is, a neutron gives up its electron and becomes a proton, and the electron is then picked up by a proton which becomes a neutron. Energy, momentum as well as charge are exchanged here, giving rise to what is called an "exchange force". However the neutron was still thought of by Heisenberg as a composite of a proton and an electron. The contradiction of this view of nucleus came from Walter Heitler and Gerhard Herzberg. They had pointed out that the spectra of diatomic molecules depend critically on whether their atomic nuclei contained an odd or an even number of elementary particles, then though protons and electrons. In a molecule with two identical nuclei, each containing an even number of elementary particles, half of the molecular energy levels that would normally be present in a pair of nonidentical nuclei are absent.. If each contains an odd number of particles, then the other half of the energy levels are absent. For example oxygen results were correct, but for nitrogen, which may contain odd number of particles in nucleus, it did not match obtained spectrum. The solution was to suppose that the neutron is an elementary particle, like the proton and the electron. Chadwick knew about this line of reasoning, but he does not seem to have taken it seriously. Near the end of 1932 he remarked[1]:

"It is, of course, possible to suppose that the neutron is an elementary particle. This view has little to recommend it at present, except the possibility of explaining the statistics of such nuclei as N^{14}."

There seems to have been a disinclination to introduce new elementary particles - a disinclination so powerful that physicists would rather consider giving up well-established physical principles, such as molecular spectra or even energy conservation than contemplate a new particle. It is therefore difficult to point the moment at which the neutron became accepted as a fully accredited elementary particle. But after analyzing forces between proton and neutrons and between two protons. It showed, that they are equal and so neutron cannot contain proton. This also denied Heisenberg theory of exchanging electron. When all properties of proton and neutron were discovered, it started to be obvious that both particles can be considered as an exhibition of one particle, which has two possible projections. It is similar to separation of two electrons with spin. Therefore that property was called isospin. This defines the new degree of freedom. This distribution can be described by $SU_{(2)}$ symmetry, where proton and neutron fits into doublet structure.

Chapter 3

The prediction and discovery of π meson

3.1 Hideki Yukawa

Figure 3.1: Hideki Yukawa[19]

born 23th of January 1907 in Tokyo Japan[8]. He graduated at Kyoto Imperial University in 1929; in 1933 he moved to Osaka Imperial University where he earned doctorate in 1938. After that he rejoined Kyoto Imperial University as a professor of theoretical physics. Later he worked at the Institute for Advanced Study in Princeton and at Columbia University. During 1935 – 1970 he was a director of the Research Institute for Fundamental Physics in Kyoto. In 1935 while a lecturer at Osaka Imperial University, he proposed a new theory of nuclear forces where he predicted the existence of mesons. The discovery of muon (originally considered to be Yukawa's meson) among cosmic rays in 1937 established Yukawa's fame as the founder of meson theory. After devoting himself to it's development, he started to work in 1947 on a more comprehensive theory of elementary particles based on his idea of the so-called nonlocal fields. He was awarded with the Nobel prize for Physics in 1949[19]. Yukawa died at 8th of September 1981 in Kyoto.

3.2 Strong nuclear force theory

After declining Heisenberg's theory of nuclear force there was a great need for any acceptable theory. Then, in 1935, a Japanese theorist Hideki Yukawa proposed a new approach[18]. The binding force must be short ranged, keeping protons and neutrons within a range of about 10^{-15}m in consequence of the size of a nucleus. According to the uncertainty principle, exchanging a particle with a mass sets a limit on the time allowed for the exchange and therefore restricts the range of the resulting force. He proposed that the mass of exchanging particle is inversely proportional to the interaction range. Yukawa had the courage to propose a new kind of charged particle with a mass two hundred times larger than that of electron, whose exchange produces a nuclear force with a range of the order of the observed nuclear size 10^{-15}m. Because the predicted mass of the new particle was between those of the electron and the proton, the particle was named mesotron (from the Greek word meso=middle,between), later shortened to meson. Yukawa's work was little known outside Japan until 1937, when Carl P. Anderson, Seth H. Neddermeyer, C. E. Stevenson and J. C. Street[1] announced the discovery of a new particle in cosmic rays with the mass exactly equal 200 electron masses. It was widely assumed at that time that it was the meson predicted by Yukawa (it was thanks to J. Robert Oppenheimer and Robert Serber[18] who made Yukawa's work more widely known in the west). In the following years, it became clear that there were difficulties in reconciling the properties expected for Yukawa's intermediary particle with those of the new cosmic ray particle. In 1945 an experiment by M. Conversi, E. Pancini and O. Piccioni[1] demonstrated the fact that the cosmic ray particles penetrate matter far too easily and therefore interact weakly with neutrons and protons. To resolve this paradox, theorists from Japan - S. Sakata and T. Inoue and independently Hans A. Bethe and R. E. Marshak[1] from United States proposed the existence of two mesons. It suggests that heavier Yukawa's nuclear meson decays into the penetrating meson from cosmic rays. The former was called π meson or pion, the latter one was called μ meson or muon. In 1947 C.M.G. Lattes, C.P.S. Occhialini and C.F. Powell[1] at Bristol University in England found the first experimental evidence of two mesons in cosmic rays high on the Pic du Midi in France. They registered the presented decay on a special photographic emulsion. By our definition of hadrons, muon does not belong to this group of elementary particles and so it is of no interest to us. Unfortunately in the pion theory there were some shortcomings. It was proposed that it acts only between proton and neutron[2]. In 1938 Nicholas Kemmer in England proposed that the nuclear force is charge invariant, which required existence of a neutral exchange particle similar to Yukawa's pion. It also established the concept of isospin invariance. Since the neutral variant is not electrically charged, the neutral pion is more difficult to observe than the charged pions. Its existence was inferred from its decay products in cosmic rays, a so-called "soft component" of electrons and photons. The π^0 was identified at the Berkeley cyclotron in 1950 by its decay into two photons. Pions come in three varieties: positive and negative charged $273, 1232$ times heavier than the electron and a neutral variant with a mass of $264, 1129$ times that of electron. They form an isotriplet much like the nucleon isodublet. All pions are unstable. Charged variant decays into muon and antineutrino with a lifetime of $2, 603.10^{-8}$s. The neutral pion decays into two photons in about $0, 8.10^{-16}$s.

Chapter 4

Strange particles

4.1 Discoverers

It is difficult to state one person as a leading discoverer of this stage. There were three persons mainly involved in fundamental progress in this field. First we have to present Kazuhiko Nishijima[2], a Japan physicist, who was the first to present the idea of existence of some charge in strange particle physics. Than there was a Dutch physicist Abraham Pais, who created the proper theory of strange particles. At last we have Murray Gell-Mann, who formed some great ideas in this theory. In this times there becomes difficult to present isolated person as discoverer of some theory.

4.2 Strange discoveries

In the same year when the pion was discovered (1947) Clifford Butler and George Rochester[5] from Great Britain, while studying cosmic rays, discovered the first examples of another type of new particle. It was heavier than pion but lighter than proton with a mass 800 times the electron mass. Yet it probably wasn't the first time the strange particle was observed, it was the first time it was properly interpreted. In 1943, four years before Butler and Rochester, Leprince-Ringuet observed[2] a particle 1000 times heavier than electron mass. Although it was published, no interpretation was presented. Because it was an isolated event, no one paid attention to it. Within the next few years after 1947 others strange particles were found, some of them heavier than proton. Although they were produced in strong interaction, they lived for a long time. It was expected, that they will decay into proton and pion via strong interaction. But experiments showed, that they decays via weak interaction, with no obvious reason. For they were called "strange" by Gell-Mann. By 1953 at least four different kinds of strange particles had been observed. First attempt to classify them and to explain strange decay behavior was done by Nishijima Kazuhiko[5] in Japan. He suggested a new conservation law. The same was done later by Murray Gell-Mann, who foretold that two of new particles form an isospin doublet and other two belongs to isospin triplet. They argued that the strange particles must possess some new property, named "ν-charge" by Nishijima[2] and "strangeness" by Gell-Mann, that is conserved in the strong nuclear decays but is not conserved in weak decays. Each particle is assigned a strangeness quantum number S. Great contribution was done by Dutch physicist Abraham Pais who (as Nishijima) in 1952 formulated the phenomenon of associated production[5]. Because of conservation of strangeness, the strong nuclear force can produce strange particles only in pairs with total strangeness equal zero.

Chapter 5

Quark model

5.1 Murray Gell-Mann

Figure 5.1: Murray Gell-Mann[19]

He was born 15th of September 1929 in New York USA[12]. Gell-Mann entered Yale university at the age of 15. In 1948 he moved to MIT. In 1952 he joined the Institute for Nuclear Studies at the University of Chicago, where he introduced the concept of "strangeness". In 1961 Gell-Mann and Yuval Ne'eman proposed a scheme for classifying hadrons. He called it Eightfold way with analogy to Buddha's eightfold path to enlightment and bliss[11]. He speculated the existence of fundamental particles which form hadrons. He called them quarks using that term from James Joyce's novel Finnegan Wake. In 1955 Gell-Mann joined the faculty of CALTECH in Pasadena. He was appointed Millikan professor of theoretical physics in 1967. In 1969[19] he was awarded with Nobel prize. It is commonly accepted that he is the major inventor of the quark model, although his contribution is objectionable. Nishijima, Pais, Sakata, Zweig[2] and others had equal contribution to the formulation of this theory. But Gell-Mann enforced the quark model with his authority, even he wasn't sure about it's validity. Without him, the quark hypothesis could be only a weird theory in thesis work of some Ph.D. student in CERN(Zweig). Such persons are well known and well needed. We can compare him to the contribution of Carlo Rubbia for the discovery of intermediate bosons or even Albert Einstein in the beginning of the nuclear bomb research.

5.2 Searching for symmetry

In 1962 Murray Gell-Mann and Yuval Ne'eman[13], an Israeli scientist, independently showed that all known hadrons can be grouped into sets which describes certain symmetry. When we want to classify such group of particles, it is crucial to choose properties by which it will be grouped into categories. If we make a figure of such categories in the I-S plane, we obtain figures which are exact copy of $SU_{(3)}$ multiplets. For all hadrons have not been discovered in that times, these hadrons multiplets were not all completed. Great success of this approach was the fact that it could be guessed what properties will new particles have. This is the way how Gell-Mann and Ne'eman found the analogy between $SU_{(3)}$ multiplets and hadrons. They were even able to foretold the discovery of particle Ω^-, which was observed in Brookhaven. As we know from the group theory (see appendix) if we can describe something with group theory it suggests some kind of internal symmetry. The beauty of $SU_{(3)}$ symmetry does not, however, explain, why it holds true. In 1964 Gell-Mann and George Zweig[13] independently decided that this symmetry has to lie in the fundamental nature of hadrons. The simplest nontrivial representation of $SU_{(3)}$ has three elements, from which we can construct all multiplets of $SU_{(3)}$ by means of the decomposition of tensor product. That says all states which represent hadrons can be composed of three fundamental states. This nontrivial representation is crucial for $SU_{(3)}$. They both made suggestion that the hadrons were not simple structures, but were instead built from three basic particles, which corresponds to that nontrivial representation. Zweig called them "aces" and Gell-Mann "quarks"[2]. We will stick to the name quark. For this to be possible,quarks had to have some unusual properties. That was the reason everyone, except Zweig, talked about them as a useful mathematical fiction. But through years there were some indirect proves for the existence of quarks. So in 1964 Gell-Mann and Zweig required only three quarks to build all known particles. These were called by Gell-Mann up, down and strange. This model provided a simple picture in which all mesons are consisting of a quark and an antiquark and all baryons as composed of three quarks. That corresponds to the tensor product of 3 and $\overline{3}$. Meson states can be achieved by decomposing tensor product into the sum of irreducible representations. In symbolic notation

$$3 \otimes \overline{3} = 8 \oplus 1$$

So each meson is described as a state of octet or singlet. Baryon states can be achieved by decomposing

$$3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$$

So each baryon is described as a state of a decuplet, octet or singlet. The up quark has the charge of $\frac{2}{3}$ and down and strange quark has the charge of $-\frac{1}{3}$. Each has baryon number $\frac{1}{3}$ and spin $\frac{1}{2}$(see 8.3 and 7.6).

Chapter 6

Parton model

6.1 Electron-proton scattering - the chase for quarks

In 1955 Robert Hofstadter[2] and his group at Stanford University turned attention to the investigation of the structure of individual nucleons. He used 200MeV electrons in elastic electron-proton scattering. The results showed that the proton does not behave as a point-like source of the Coulomb field. Furthermore, the root-mean-square radius of proton $r_{ch} = r_{mag} = 0,7 \pm 0,24\ fm$ was determined from the results of that experiment. Later in 1955 the electron energy was increased to 550MeV so it could reach the momentum transferred squared up to $0,5GeV^2$ and improve the precision of measurement of the Coulomb field distribution in proton. These investigations were completed in early sixties using 1000MeV electron beam from the Mark III Linac. Success of this 1GeV Linac lead to the decision to build a new electron linear accelerator with energies up to 20GeV at Stanford Linear Accelerator Center (SLAC). Primary aim of that research was the extension of elastic scattering experiments as well as the quasi-elastic scattering (electro-production of resonances) and just for completeness the inelastic continuum. The two miles long machine was built in late 1966 and in 1967 the group of experimentalists from SLAC and MIT started a series of deep inelastic electron-proton scattering experiments, but only as an extension to elastic experiments. The first results on electron-proton were obtained in early 1968 and were reported[2] at the IV. International Conference on High Energy Physics in July of the same year. Previous effects were approved up to $Q^2 \sim 30GeV^2$, furthermore from inelastic part the first possibility that nucleon is composed of point-like charged structures came. But still no-one had any ideas how the inelastic formfactors(structure functions $F_1(x, Q^2), F_2(x, Q^2)$) depend on Q^2. Although quark model has been already formulated, it hadn't been considered to have any dynamic effects. It was just static model. Theoretical framework describing this phenomena was created mostly by James Bjorken and Kurt Gottfried[2]. This was the first step to dynamic model of hadrons.

6.2 Dynamic solution of quark model

The first MIT-SLAC results on deep inelastic scattering presented in summer of 1968 had attracted the attention of most theorists, mainly Richard P. Feynman and Bjorken. Feynman had developed the basic ideas of the parton model during several visits to SLAC in autumn 1968. He is considered to be the author of that model although he hadn't published it as first[2]. The first published paper on parton model and his application to the analysis of deep inelastic scattering of leptons on nucleons was written by Bjorken and Paschos in April 1969. The basic idea of Feynman model is to present the inelastic electron-proton scattering as quasi-free scattering from point-like constituents within the proton. Crucial for the interpretation of this proces is its description in the system of infinite proton momentum and the presumption of deep inelastic scattering. Model also contained the mechanism of hadronisation i.e. the conversion of final state partons into observable hadrons. The spin $\frac{1}{2}$ nature of charged partons immediately raised the question of their relation to the constituent quarks. Although there is a close relation between them, these two concepts are not identical. It has become generally accepted practice[2] to call Feynman's parton model the Quark Parton Model (the charged, spin $\frac{1}{2}$ partons, are usually called current quarks to emphasize the difference from constituent quarks). The mass of current ones is about 10MeV in contrast to static constituent ones, which are predicted to be about 300MeV (for u,d). Furthermore, there is no fixed number of current quarks inside proton.

6.3 Genesis of Quantum chromodynamics

In 1969 we were in situation when Feynman and Bjorken formulated the parton model as the result of experimental data from SLAC. In that model nucleon behaved in hard collisions as a beam of almost non-interacting point-like constituents. By the early 1973 the data had provided the evidence for identification of charged partons with quarks and indirect evidence for the presence of neutral partons in nucleon as well[2]. The candidates for neutral partons were the gluons introduced by Nambu in his model of interquark forces mediated by the exchange of octet of "colored" vector bosons. By 1973 most important parts of complete strong interaction theory were invented. The last step was to show that the quark-parton model follows from some local field theory. The main influence on the final formulation had David Gross and Frank Wilczek on 24th of April and independently David Politzer on 3rd of May[2]. The crucial step was the formulation of the property of asymptotic freedom. The full formulation of Quantum chromodynamics was published in Physical Review at the end of July 1973. Major results of this theory can be summarized as following. Quarks in hadrons are bound together by exchanging gauge bosons called gluons. The quarks carry a property called color that is analogous to electric charge. So every colored particles "feel" strong interaction and exchange gluons. This also applies to gluons themselves. Gluons are massless and have the spin of 1. There are three types of color charge called blue,green and red followed by anticolors. Quarks each carry a single color charge, while gluons carry both a color and anticolor charge. The strong force acts in such a way that quarks of different color are attracted to one another but those of the same color repel each other. The quarks can combine only in ways that give a net color charge of zero. The color of quark and the anticolor of antiquark cancels each other. The main property of color theory is that no color charged particles can be seen in unbound state. Therefore we cannot see free quarks or gluons.

Chapter 7

More quarks on scene

7.1 Charm quark,SU(4)

In late 1960s after the GWS theory was formulated, there became a problem with combining it with Gell-Mann's model of three quarks[6]. From the beginning of GWS model it was obvious that an application of local gauge symmetry $SU_{(2)} \times U_{(1)}$ to weak and electromagnetic interactions between quarks u,d,s yields to direct interaction of quarks d,s and neutral boson Z^0. So there is a soft neutral current in which the strangeness does change. Major influence of this conclusion can be seen on K^- decay because

$$
\begin{aligned}
K^- &\rightarrow \pi^- \, e^+ \, e^- \\
K^- &\rightarrow \pi^0 \, e^- \, \overline{\nu}
\end{aligned}
$$

shall be of the same frequency. This is in conflict with experimental data. In 1970 Sheldon Glashow, John Iliopoulos and Luciano Maiani[6] proposed a four quark model which was electroweak local symmetry compatible. The prediction also contained properties of new quark - charge $\frac{2}{3}$ and new quantum number called "charm" which is conserved in strong interaction but not in weak interaction. This quark was called "charmed" by Bjorken and Glashow and denoted c. The rest mass was predicted to be $1,5$GeV. But that still was only theoretical scheme. In autumn of 1974 there was an interesting discovery. As usual it was made by two groups independently. At SLAC in Stanford under Burton Richter a resonance was observed at $3,1$GeV with surprisingly small width(keV) in colliding beams of electrons and positrons. They called this particle ψ[23]. At Brookhaven under Samuel Ting a resonant structure at $3,1$GeV was identified in proton-Beryllium interactions. They called it J[25]. The resonance was found in the electron-positron decay channel. They informed themselves in 11.11.1974 and published results in Physical Review Letters calling the new meson J/ψ. Ten days after the first discovery Richter's group identified[24] another resonance called ψ' at $3,7$GeV. It was obvious, that both are bound states of $c\bar{c}$.

After this discovery, quarks were widely accepted as physical entities. It has to be said that J/ψ and ψ' are called "hidden charm" particles because both are $c\bar{c}$ and therefore they possess the charm 0. The next step was to find "overt charm" particles. That was done at Stanford in 1976. The new mesons were D^+ and D^0, which consists of $c\overline{d}(c\overline{u})$. Richter and Ting obtained Nobel prize[19] for their discoveries in 1976. With the discovery of fourth quark the $SU_{(3)}$ flavor symmetry was extended into $SU_{(4)}$ one. Multiplets of $SU_{(4)}$ were constructed in three dimensions $Y - T_3 - C$ space where C denotes charm quantum number (see figs7.5,7.6). There are three Casimir operators so the rank is equal 3.

$$\begin{aligned} 4 \otimes 4 \otimes 4 &= 20 \oplus 20 \oplus 20 \oplus 4 \\ 4 \otimes \overline{4} &= 15 \oplus 1 \end{aligned}$$

Gell-Mann Nishijima equation has the form of

$$Q = T_3 + \frac{Y + C}{2}$$

7.2 Bottom quark,SU(5)

Although it seemed that the symmetry in lepton and quark families were restored, in 1975 there became a problem again by discovering τ lepton. There is a serious reason for the symmetry between the number of quarks and leptons. It is the inner consistence of perturbative expansion in GWS theory (and also the ability to renormalize Feynmann diagrams in high orders of perturbative expansion)[6]. So until 1977 there was an intensive search for other two quarks. In 1977 at FNAL(Fermilab) the team led by Leon Lederman[2] found fifth quark. They were studying the production of lepton pairs in hadronic collisions and found two resonance peaks in spectrum of $\mu^+\mu^-$ pairs produced in collisions of 400GeV protons with nuclei. Invariant masses were $9,5$GeV and 10GeV and new particles were signed Υ and Υ'(upsilon)[26]. It was again obvious that they form a bounded state of a new quark denoted b(bottom). The bottom quark has the mass of $4,5$GeV and charge $-\frac{1}{3}$ and presents new quantum number "beauty". It also extended quark model which is now described by $SU_{(5)}$.

7.3 Top quark

The existence of the sixth quark (which would complete the third generation of quarks) was widely expected after the discovery of the bottom quark. It can be noted that many theorists tried to derive its mass from existing theories but all failed[2] before the LEP results. From precise measurement of Z^0 and $W^\pm$ bosons properties followed that the mass of top quark is in the range of $170-180$GeV. Top quark was first discovered in two experiments D0[27] and CDF[28] at FNAL on Tevatron in proton antiproton collisions with energy $1,8$TeV and it was confirmed next year. The rest mass was determined as 174 ± 5GeV and therefore it is the heaviest elementary particle known. The mean lifetime of t quark was determined as 10^{-25}s and so we can ask whether it is an elementary particle because it decays faster than it can form any bound state. This is the reason, why the hadron spectrum cannot be described by $SU_{(6)}$ multiplets. Top quark holds a new quantum number "true". Therefore the final quark model symmetry is described by $SU_{(6)}$. Generalized Gell-Mann Nishijima equation is

$$Q = T_3 + \frac{Y + C + B + T}{2}.$$

Multiplets of $SU_{(6)}$ are five dimensional objects in $Y - T_3 - C - B - T$ space.

$$\begin{aligned} 6 \otimes 6 \otimes 6 &= 56 \oplus 70 \oplus 70 \oplus 20 \\ 6 \otimes \overline{6} &= 35 \oplus 1 \end{aligned}$$

7.4 Quark properties

The properties of all quarks are summarized in the following table

Name	Symbol	Mass	Q	I_3	S	C	B	T
down	d	$3 \sim 9MeV$	$-\frac{1}{3}$	$-\frac{1}{2}$	0	0	0	0
up	u	$1.5 \sim 5MeV$	$+\frac{2}{3}$	$+\frac{1}{2}$	0	0	0	0
strange	s	$60 \sim 170MeV$	$-\frac{1}{3}$	0	-1	0	0	0
charmed	c	$1.47 \sim 1.83GeV$	$+\frac{2}{3}$	0	0	+1	0	0
bottom	b	$4.6 \sim 5.1GeV$	$-\frac{1}{3}$	0	0	0	-1	0
top	t	$178.1^{+10.4}_{-8.3}GeV$	$+\frac{2}{3}$	0	0	0	0	+1

Figure 7.1: Summary table of quark properties

7.5 Pentaquarks

In the formulation of quark model there was postulated that baryons consist of three quarks and mesons of quark antiquark pair. This is the simplest way to satisfy all hadron properties. But soon there arose questions whether there can exist "exotic" hadrons. Mainly exotic baryons composed of four quarks and an antiquark $qqqq\overline{q}$ called pentaquarks and exotic mesons composed of two quarks and two antiquarks $q\overline{q}q\overline{q}$ called tetraquarks. No theory forbids such compound, even QCD doesn't. Physicists have searched for a five-quark state for more than 35 years. In 1997 Dmitri Diakonov, Victor Petrov and Maxim Polyakov predicted from the chiral soliton model an exotic isoscalar antidecuplet of five-quark resonances.

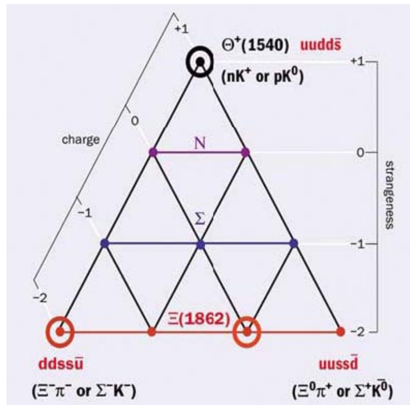

Figure 7.2: Pentaquark decuplet, three of particles with exotic-flavour quantum numbers [34]

In recent years four laboratories (LEPS in Osaka Japan, Jefferson Lab in Ohio US,ITEP in Moscow Russia and ELSA in Bonn Germany.)[29][33] presented a discovery of a baryon bound state with antiquark $\bar{s}$. Therefore such baryon has to be pentaquark with predicted composition $uudd\bar{s}$ called Θ^+ with a mass of 1540MeV. Next indirect proof is the Θ^+ strong decay into neutron and K^+. Several groups of theorists formed theories describing such quark multiplets. Some of them presented a model of Θ^+ composed of $(ud)(ud)\bar{s}$, where (ud) are correlated doublequarks, or model with "molecular" bound state of meson and baryon. But in spite of that all this phenomena remain objectionable, because some physicists and also laboratories which presented the discovery started to annulate that discovery[32],[31],[30].

7.6 Final conclusion - present state

After we have reviewed history of hadron discoveries, let's associate particles with algebraic structure of group theory. In the chapter 8.3 the connection between particle multiplets and multiplets of $SU_{(N)}$ is presented. Each particle multiplet can be described by its maximum weight states $(T_3)_{max}$ and $(Y)_{max}$ and $SU_{(N)}$ multiplet can be uniquely characterized by two numbers p,q. But two particle multiplets can have the same shape, although they contain different set of particles. So this correspondence is not unique. It suggests the demand of some degrees of freedom to differ such multiplets. We can use total angular momentum J and parity of particle P. So each particle multiplet is uniquely characterized by two maximum weight states and two "inner" degrees of freedom J^P. Although from the view of group theory, two particle multiplets with the same maximum weight states are identical. As an example of two different particle multiplets described by the same $SU_{(N)}$ multiplet are octets of pseudoscalar 0^- and vector 1^- mesons. Some $SU_{(3)}$ and $SU_{(4)}$ particle multiplets for mesons and baryons are shown in figs7.3,7.4,7.5,7.6.

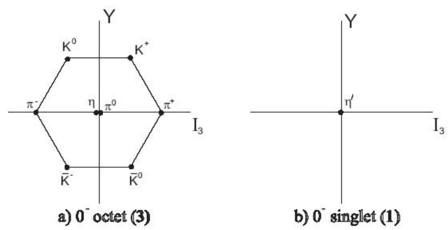

Figure 7.3: Pseudoscalar meson octet and singlet

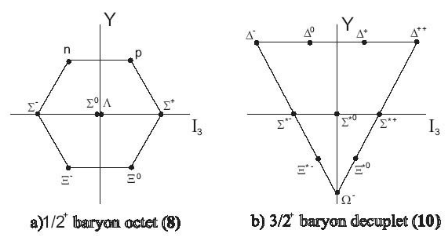

Figure 7.4: Baryon octet and decuplet

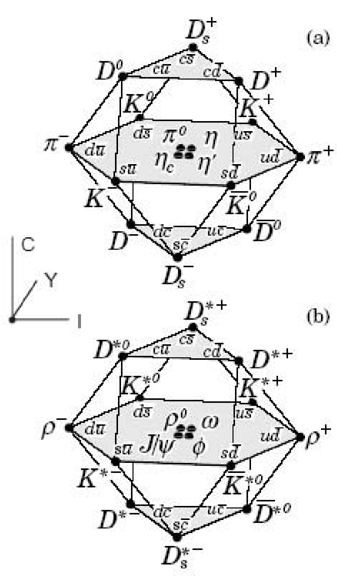

Figure 7.5: $SU_{(4)}$ meson states

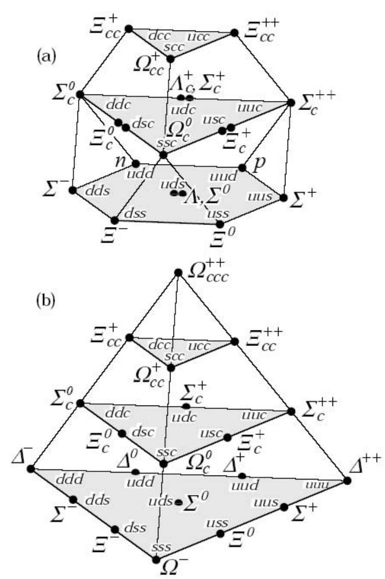

Figure 7.6: $SU_{(4)}$ baryon states

In the present state of knowledge we distinguish about 240[7] mesons and their excited states and about 135[7] baryons and their excited states. The hadronic spectrum is described by multiplets of $SU_{(5)}$ symmetry group.

Mesons		Baryons	
light unflavoured(S=C=B=0)	121	N baryons(S=0,I=$\frac{1}{2}$)	15
strange(S=±1,C=B=0)	51	Δ baryons(S=0,I=$\frac{3}{2}$)	40
charmed(C=±1)	18	exotic baryons	1
charmed,strange(S=C=±1)	12	Λ baryons(S=-1,I=0)	14
bottom(B=±1)	7	Σ baryons(S=-1,I=1)	30
bottom,strange(B=S=±1)	3	Ξ baryons(S=-2,I=$\frac{1}{2}$)	12
bottom,charmed(B=C=±1)	1	Ω baryons(S=-3,I=0)	2
hidden charmed(C=±0)	14	charmed baryons(C=+1)	20
hidden bottom(B=±0)	13	bottom baryons(B=-1)	1

Chapter 8

Mathematical framework of Lie groups

8.1 General introduction into group theory

Let's review some of well-known facts about groups.

Definition 1 *Some non-empty set* $\mathbb{G}$*,which has a binary operation* $\star : \mathbb{G} \times \mathbb{G} \to \mathbb{G} \quad (a, b) \to a \star b$ *defined and which meets requirements:*

(1) $\forall a, b, c \in \mathbb{G} \qquad a \star (b \star c) = (a \star b) \star c$

(2) $\exists \mathfrak{o} \in \mathbb{G}$ *(neutral element)* $\forall a \in \mathbb{G} \qquad (\mathfrak{o} \star a = a)$

(3) $\forall a \in \mathbb{G} \quad \exists b \in \mathbb{G}$ *(inverse element)* $\qquad (a \star b = \mathfrak{o})$

is called <u>*group*</u>*. If the binary operation is addition(multiplication), we call such group additive(multiplicative). If the group fulfills the commutative law, it is called Abelian.*

Definition 2 *The group* $\mathbb{G}$*,on which is defined a binary operation* $[,] : \mathbb{G} \times \mathbb{G} \to \mathbb{G}$ *and which meets requirements:*

(1) $\forall a, b \in \mathbb{G} \qquad [a, b] = -[b, a]$

(2) $\forall a, b, c \in \mathbb{G} \quad [a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$

is called <u>*algebra*</u>*.*

Definition 3 *We will call a group* <u>*continuous*</u>*, if its elements are functions of one or more continuous variables*

$$G = \{a_{(t)}, b_{(t)}, \ldots\}$$

where t is of continuous set.

Definition 4 *We will call a group continuously connected, if a continuous variation of the group parameters leads from any arbitrary element of the group to any other.*

Definition 5 *Let's have two groups G and G' and a transformation between elements $f : G \to G'$.*

(1) f is called homomorphism $\Leftrightarrow \forall g_1, g_2 \in G(f_{(g_1 \star g_2)} = f_{(g_1)} \star' f_{(g_2)})$. *The two groups are called homomorphic.*

(2) f is called isomorphism $\Leftrightarrow$ *f is bijective homomorphism and hence f^{-1} exists. The two groups are called isomorphic.*

Let's have two algebras A and A', which are vector spaces on the number field K with an inner product $[,]_A$ ($[,]_{A'}$) and a transformation between elements $f : A \to A'$.

(1) f is called homomorphism $\Leftrightarrow \forall a_1, a_2 \in A \forall \alpha_1, \alpha_2 (f_{(\alpha_1 a_1 + \alpha_2 a_2)} = \alpha f_{(a_1)} + \alpha_2 f_{(a_2)})$ *and* $f_{([a_1,a_2]_A)} = [f_{a_1}, f_{a_2}]_{A'}$ *The two algebras are called homomorphic.*

(2) f is called isomorphism $\Leftrightarrow$ *f is bijective homomorphism and hence f^{-1} exists. The two algebras are called isomorphic.*

Definition 6 *If a group G is isomorphic to the space of linear operators on Hilbert space $\mathcal{L}_{\mathcal{H}}$, than the value domain of the transformation is called representation of G. Especially if we write operators as their matrixes in arbitrary but fixed basis, they form a group and it is called matrix representation of G*

Definition 7 *A subset $\mathbb{P} \subseteq \mathbb{G}$,which is closed under the binary operation of group $\mathbb{G}$ and under unitary operation inverse is called subgroup of group $\mathbb{G}$.*

Definition 8 *The subgroup $\mathbb{P}$ is called normal(invariant) subgroup if $\forall p \in \mathbb{G}$ $g.\mathbb{P}.g^{-1} = \{g.p.g^{-1} | p \in \mathbb{P}\} = \mathbb{P}$ holds.*

Definition 9 *The group $\mathbb{G}$ is called semisimple, if it does not posses any invariant Abelian subgroup. The group is called simple, if it does not posses any invariant subgroup.*

Definition 10 *Continuous group, whose elements are given by operators $\hat{U}_{(\alpha_1,\alpha_2,\ldots,\alpha_n,r)}$, which depend on* **n** *real parameters, are called Lie group.*

We demand differentiability of $\hat{U}$ with respect to α_i $\forall i \in \hat{n}$. The symbol $\hat{n}$ means a set $\{1, 2, \ldots, n\}$ from now. The argument r stands symbolically for a possible coordinate dependence. It is advantageous to choose the parameters such that $\hat{U}_{(\vec{\mathbf{0}})} = \mathbb{1}$ holds.

Theorem 11 *One can represent the operators of the group in the form*

$$\hat{U}_{(\alpha_1,\alpha_2,\ldots,\alpha_n,r)} = e^{-i\sum_{\mu=1}^{n} \alpha_\mu \hat{L}_\mu},$$

where

$$\hat{L}_\mu = i\frac{\partial \hat{U}_{(\vec{\alpha})}}{\partial \alpha_\mu}|_{\vec{\alpha}=\vec{0}}$$

are operator functions called generators of the group.

Proof 1 *For infinitesimal transformation in the neighbourhood of the identity*

$$\hat{U}_{(\delta\alpha_\mu)} = \hat{U}_{(0)} + \frac{\partial \hat{U}_{(\vec{\alpha})}}{\partial \alpha_\mu}|_{\vec{\alpha}=\vec{0}}.\delta\alpha_\mu = \mathbb{1} - iL_\mu \delta\alpha_\mu$$

If we perform finite transformation α_μ composed of N infinitesimal transformations, we obtain

$$\hat{U}_{(\alpha_\mu)} = \lim_{N\to+\infty}[\hat{U}_{(\delta\alpha\mu)}]^N = \lim_{N\to+\infty}[\mathbb{1}-iL_\mu\delta\alpha_\mu]^N = \lim_{N\to+\infty}[\mathbb{1}-iL_\mu\frac{\alpha_\mu}{N}]^N = e^{-iL_\mu\alpha_\mu}$$

We have made use of a group property, that we can construct a finite group element from the product of infinitesimal elements. For more variables we obtain

$$\hat{U}_{(\delta\vec{\alpha})} = \hat{U}_{(\vec{0})} + \sum_{\mu=1}^{n} \frac{\partial \hat{U}_{(\vec{\alpha})}}{\partial \alpha_\mu}|_{\vec{\alpha}=\vec{0}}.\delta\alpha_\mu = \mathbb{1} - i\sum_{\mu=1}^{n} L_\mu \delta\alpha_\mu$$

$$\hat{U}_{(\vec{\alpha})} = \lim_{N\to+\infty}[\hat{U}_{(\delta\vec{\alpha})}]^N = \lim_{N\to+\infty}[\mathbb{1} - i\sum_{\mu=1}^{n} L_\mu\delta\alpha_\mu]^N =$$

$$\lim_{N\to+\infty}[\mathbb{1} - i\sum_{\mu=1}^{n} L_\mu\frac{\alpha_\mu}{N}]^N = e^{-i\sum_{\mu=1}^{n} L_\mu\alpha_\mu} \ldots \mathbf{Q.E.D}$$

Theorem 12 *$\{\hat{L}_\mu\}$ has to be linearly independent (that says $\sum_{i=1}^{n}\hat{L}_i\delta\alpha_\mu = 0 \Leftrightarrow \delta\alpha_i = 0 \forall i \in \hat{n}$)*

Theorem 13 *If $\hat{U}_{(\alpha_\mu)}$ is unitary ($\hat{U}^\dagger_{(\alpha_\mu)} = \hat{U}^{-1}_{(\alpha_\mu)}$), than the generators has to be hermitian.*

Proof 2 *From further analysis of operator properties comes, that $\hat{U}^{-1}_{(\delta\alpha_\mu)} = \hat{U}_{(-\delta\alpha_\mu)}$*

$$\hat{U}^\dagger_{(\delta\alpha_\mu)} = \mathbb{1} + i\sum_\mu \delta\alpha_\mu \hat{L}^\dagger_\mu$$

We have chosen real parameters $\delta\alpha_\mu$ in definition of group.

$$\hat{U}^{-1}_{(\delta\alpha_\mu)} = \mathbb{1} + i\sum_\mu \delta\alpha_\mu \hat{L}_\mu$$

$$\Rightarrow \quad \sum_\mu \delta\alpha_\mu(\hat{L}^\dagger_\mu - \hat{L}_\mu) = 0$$

and $\delta\alpha_\mu$ are linearly independent $\Rightarrow \hat{L}^\dagger_\mu = \hat{L}_\mu \quad \forall\mu \in \hat{n}$

Last implication comes from general law of variational analysis.

Theorem 14 *Group generators have to satisfy commutation relations* $[\hat{L}_k, \hat{L}_m] = c_{kmj}\hat{L}_j$ *and therefore form a closed commutator algebra. Constant numbers* c_{kmj} *are called structure constants.*

Theorem 15 *Structure constants* c_{ijk} *are antisymetric in the first two indices* ($c_{ijk} = -c_{jik}$).

Theorem 16 *This relation holds for structure constants*

$$c_{ijm}c_{mkn} + c_{ikm}c_{min} + c_{kim}c_{mjn} = 0$$

.

Relations $[\hat{L}_k, \hat{L}_m] = c_{kmj}\hat{L}_j$ and $c_{ijm}c_{mkn} + c_{ikm}c_{min} + c_{kim}c_{mjn} = 0$ form the fundamental relations of the Lie algebra, which is characteristic of the group. The structure constants contain all of the information concerning the group.

8.1.1 Interpretation of commutators

The commutation relations can be viewed as a direct generalization of the vectorial cross product

$$\hat{L}_i \times \hat{L}_j = c_{ijk}\hat{L}_k$$

This is generalized relation $e_i \times e_j = e_k$ in the three-dimensional vector space. Note that it says, we can obtain "basis vector" as a cross product of other "basis vectors." Our course of derivation started from Lie group. Then we determined its generators and then calculated the commutators. In this way we were led to the Lie algebra. Although, it can be done in reversed order, which is described by Lie's theorem.

Theorem 17 (Lie's theorem) *If a set of N hermitian operators* $\hat{L}_i$ *is given, which is closed under commutation, than these operators* $\hat{L}_i$ *specify a Lie group, whose generators they are.*

An essential characteristic of a Lie group is its rank.

Definition 18 *The largest number of generators commuting with each other is called rank.*

For example the translation group (Abelian) has three generators $\hat{p}_\nu = -i\frac{\partial}{\partial x \nu}$, which all commute with each other, and hence the rank is 3. Rotation group SO(3) has rank 1 and SU(3) has rank 2.

Definition 19 *Lie group is called simple, if it does not posses continuous invariant subgroup. Lie group is called semisimple, if it does not posses a continuous Abelian invariant subgroup(still it can have non abelian one). For Lie group holds, that it can contain discrete invariant subgroup and still be simple or semisimple.*

Let's look at semisimple group:

$$\hat{a}_j = \hat{g}_\nu \hat{a}_i \hat{g}_\nu^{-1} \qquad / \hat{a}_i^{-1} \leftarrow$$
$$\hat{a}_l := \hat{a}_j \hat{a}_i^{-1} = \hat{g}_\nu \hat{a}_i \hat{g}_\nu^{-1} \hat{a}_i^{-1}$$

Since $\mathbb{G}$ and $\mathbb{A}$ are Lie groups, we call $\hat{G}_k$ the operators belonging to $\mathbb{G}$ and $\hat{A}_j$ generators belonging to $\mathbb{A}$. So

$$\begin{aligned} \hat{g}_\nu &= \mathbb{1} - i\delta\alpha_i \hat{G}_i - \tfrac{1}{2}\delta\alpha_i\delta\alpha_j \hat{G}_i \hat{G}_j \\ \hat{a}_i &= \mathbb{1} - i\delta\beta_k \hat{A}_k - \tfrac{1}{2}\delta\beta_k\delta\beta_l \hat{A}_k \hat{A}_l \end{aligned}$$

$$\begin{aligned} \hat{a}_l = \hat{g}_\nu \hat{a}_i \hat{g}_\nu^{-1} \hat{a}_i^{-1} \quad &= \ldots = \mathbb{1} - \delta\alpha_i \delta\beta_k [\hat{G}_i, \hat{A}_k] \\ \hat{a}_l = \mathbb{1} - i\delta\gamma_m \hat{A}_m & \end{aligned}$$

$$\Rightarrow \qquad i\delta\gamma_m \hat{A}_m = \delta\alpha_i\delta\beta_k [\hat{G}_i, \hat{A}_k]$$

Let's have $i\delta\gamma_m = a_{ikm}\delta\alpha_i\delta\beta_k$, so

$$[\hat{G}_i, \hat{A}_k] = a_{ikl}\hat{A}_l \quad \forall \hat{G}_i$$

Therefore, if one can linearly combine M generators $\hat{A}_l$ ($M < N$) out of N generators $\hat{G}_i$ of a Lie group, so that $[\hat{G}_i, \hat{A}_k] = a_{ikl}\hat{A}_l$ holds, than the Lie group possesses an invariant subgroup. The M generators $\{\hat{A}_1 \ldots \hat{A}_M\}$ of the invariant subgroup form a subalgebra of the original Lie algebra. Such subalgebra is called the ideal.

Theorem 20 *Lie algebra is simple if it does not posses an ideal apart from the null ideal $\{0\}$. Lie algebra is semisimple if it does not posses an abelian ideal.*

8.1.2 Cartan's criterion for semisimplicity

We define the symmetric tensor

$$g_{\sigma\lambda} = g_{\lambda\sigma} = C_{\sigma\rho\tau}C_{\lambda\tau\rho}$$

which is called a metric tensor(Killing form). The C_{ijk} are structure constants of a group. It can be defined for any Lie group and its associated Lie algebra. The metric can be defined as $\rho(\hat{L}_i, \hat{L}_j) = tr(\hat{L}_i, \hat{L}_j)$, where $\hat{L}_i$ are generators. We can take regular representation of Lie group in which the matrix elements of $\hat{L}_i$ are $(\hat{L}_i)_{\alpha\beta} = C_{i\alpha\beta}$.

$$\rho(\hat{L}_i, \hat{L}_j) = tr(\hat{L}_i, \hat{L}_j) = \sum_{\alpha\beta} (\hat{L}_i)_{\alpha\beta}(\hat{L}_j)_{\beta\alpha} = C_{i\alpha\beta}C_{i\beta\alpha} = g_{ij}$$

$\rho(\hat{L}_i, \hat{L}_j)$ also fulfills all properties of a metric, although, it is not positive definite.

Theorem 21 (Cartan's criterion) *Lie algebra is semisimple* $\Leftrightarrow det(g_{\sigma\lambda}) \neq 0$

Definition 22 *Lie group is called compact if its parametrization consists of a finite number of topologically bounded parameter domains. Otherwise it is called non compact. The same applies for related Lie algebra.*

Theorem 23 *Compact Lie group is semisimple.*

8.1.3 Casimir operators

Definition 24 *Operator* $\hat{J}$ *is called Casimir operator (invariant operator) of the group if it commutes with all generators (therefore with all group operators).*

Theorem 25 (Racah's theorem) *For any semisimple Lie group of rank* **l***, there exists a set of* **l** *Casimir operators. These are functions of the generators* $\hat{L}_i$ *(*$\hat{C}_\lambda(\hat{L}_1 \ldots \hat{L}_n)$*) and commute with every operator of the group and therefore also amongst themselves. The eigenvalues of the* $\hat{C}_\lambda$ *uniquely characterize the multiplets of the group.*

Definition 26 *Let's have an subspace of the total Hilbert space. We call it invariant if it is closed under application of group operators. That is a set of states which reproduce themselves by application of some operator of the group. The operators of the group transform the states of the invariant subspace among themselves.*

That also says that matrix elements of the group operators between states of the invariant subspace and states outside of it vanish

Definition 27 *A multiplet is an irreducible invariant subspace of a group(subspace which does not contain a further invariant subspace).*

In terms of the group theory, a set of degenerate states is called a multiplet. The multiplet depend on symmetry group. Each group has a well-defined, unique and partly characteristic set of multiplets. Although these multiplets are determined by the structure of the group, there exists no general method to find them for arbitrary continuous groups. Only for semisimple Lie group, we can use the Racah theorem.

8.1.4 Invariance under symmetry group

Let $\hat{U}_{(\vec{\alpha})}$ be arbitrary operator of a symmetry group. The invariance of the system under the group U means that both the initial state ψ and the state $\psi' = \hat{U}_{(\vec{\alpha})}\psi$ generated by the symmetry operation fulfill the same Schrödinger equation with the same Hamiltonian

$$\begin{aligned}
i\frac{\partial}{\partial t}\psi &= \hat{H}\psi \qquad /_{\rightarrow} * \hat{U}_{(\vec{\alpha})} \nsim t!!! \\
i\frac{\partial}{\partial t}\psi' &= \hat{H}\psi' \\
i\frac{\partial}{\partial t}\hat{U}_{(\vec{\alpha})}\psi &= \hat{U}_{(\vec{\alpha})}\hat{H}\hat{U}^{-1}_{(\vec{\alpha})}\hat{U}_{(\vec{\alpha})}\psi \\
\Rightarrow \underline{\hat{U}_{(\vec{\alpha})}\hat{H}\hat{U}^{-1}_{(\vec{\alpha})}} &\Leftrightarrow \quad [\hat{U}_{(\vec{\alpha})}, \hat{H}] = 0
\end{aligned}$$

The invariance of the system under the group U necessarily means that $\hat{H}$ commutes with all group operators $\hat{U}_{(\vec{\alpha})}$, and, therefore that it also commutes with all generators of the group

$$[\hat{L}_i, \hat{H}] = 0 \quad \Rightarrow \quad \hat{H}\ is\ Casimir\ operator\ for\ symetry\ group$$

Whenever a system is in eigenstate of Hamiltonian

$$\hat{H}\psi_0 = E_0\psi_0$$

then

$$\hat{U}_{(\vec{\alpha})}\hat{H}\psi_0 = \hat{U}_{(\vec{\alpha})}E_0\psi_0 \quad \wedge \quad \hat{H}\hat{U}_{(\vec{\alpha})} = \hat{U}_{(\vec{\alpha})}\hat{H} \quad \Rightarrow \quad \hat{H}(\hat{U}_{(\vec{\alpha})}\psi_0) = E_0(\hat{U}_{(\vec{\alpha})}\psi_0)$$

So all other states $\hat{U}_{(\vec{\alpha})}\psi_0$ of the multiplet are eigenstates of Hamiltonian with the same eigenvalue E_0 (Hamiltonian is degenerate on each multiplet of symmetry group). This also holds for other Casimir operators. In other words, for a given multiplet the operators $\hat{C}_\lambda$ possess a common set of eigenvalues $C_1 \ldots C_l$. Thus the Racah theorem guarantees that each multiplet is related uniquely to a set of eigenvalues $C_1 \ldots C_l$.

8.1.5 Construction of Casimir operators

There is no general way to construct Casimir operators for arbitrary group. We have to analyze each group separately.

Theorem 28 *For $SU_{(n)}$, the Casimir operators have to be simple homogenous polynomials in the generators*

$$\hat{C}_\lambda = \sum_{ij} a^\lambda_{ij} \ldots \hat{L}_i\hat{L}_j \ldots$$

where a^λ_{ij} are functions of the structure constants.

Theorem 29 *Casimir operators are not unique. If $\hat{C}', \hat{C}$ are Casimir operators, than $\hat{C} \pm \hat{C}'$ are Casimir operators. Also $\hat{C}^\alpha$ and $\hat{C}^\alpha\hat{C}'^\beta$ are Casimir operators.*

Theorem 30 *If the group operators are unitary and hence the generators $\hat{L}_i$ are hermitian, than one can always construct the Casimir operators of a unitary semisimple Lie group as hermitian operators.*

Proof 3 *$\hat{C}$ is invariant operator $\Rightarrow$*

$$\hat{C}\hat{U}_{(\vec{\alpha})} = \hat{U}_{(\vec{\alpha})}\hat{C} \quad \wedge \quad \hat{C}^\dagger \hat{U}^\dagger_{(\vec{\alpha})} = \hat{U}^\dagger_{(\vec{\alpha})}\hat{C}^\dagger$$

From unitarity of $\hat{U}_{(\vec{\alpha})}$ comes $\hat{U}^\dagger_{(\vec{\alpha})} = \hat{U}^{-1}_{(\vec{\alpha})}$.

So $\hat{C}^\dagger$ commutes with $\hat{U}^{-1}_{(\vec{\alpha})}$, which is by definition also a group element. That means $\hat{C}^\dagger$ commutes with all group operators. Obviously operator $\hat{C}' = \hat{C} + \hat{C}^\dagger$ is hermitian and therefore for all Casimir operators we can pass to new set of Casimir operators, which are hermitian.

Q.E.D

Theorem 31 *One of the Casimir operators is always given by*

$$\hat{C}_1 = g^{\rho\sigma}\hat{L}_\rho \hat{L}_\sigma$$

where $g^{\rho\sigma}$ is inverse metric tensor (for semisimple group always exists)

Theorem 32 *For Abelian Lie group it's rank is equal to it's number of generators $\hat{L}_i$. These are invariant operators themselves and therefore Casimir operators.(So Racah theorem can be extended to all Abelian Lie groups)*

Although the **l** Casimir operators are not uniquely determined, they form a complete set:

Theorem 33 (Completeness relation) *Each operator $\hat{A}$ which commutes with all operators of a Lie group is necessarily a function of the Casimir operators $\hat{C}_\lambda$ of the group*

$$\hat{A} = \hat{A}_{(\hat{C}_\lambda)}$$

(So Casimir operators are the largest set of independent operators, which commute with the group operators)

So if a system has certain symmetry, than the corresponding Hamiltonian must commute with the generators and therefore with Casimir operators. Although, it means that $\hat{H}$ itself has to be built up from invariant operators of symmetry group.

8.1.6 The connection between coordinate and function transformations

A the beginning of this chapter, we had stated, that every operator of group can be written in the form

$$\hat{U} = e^{-i\sum \alpha_\mu \hat{L}_\mu}.$$

Now we will try to prove it more exactly.
Let's have a Lie group consisting of transformation s of coordinates $x_i \rightarrow x'_i$

$$\vec{x}' = \vec{f}_{(\vec{x},\vec{a})}$$

where $\vec{x}$ and $\vec{x}'$ are n-dimensional space vectors and $\vec{a}$ represents the **r** group parameters. The parameters are chosen such that $\vec{a} = \vec{0}$ yields to identity

$$\vec{x} = \vec{f}_{(\vec{x},\vec{0})}$$

If we perform an infinitesimal transformation $d\vec{a}$ from $\vec{x}$ to $\vec{x}' = \vec{x} + d\vec{x}$, than

$$\vec{x} + d\vec{x} = \vec{f}_{(\vec{x},d\vec{a})} \qquad \vec{x} = \vec{f}_{(\vec{x},\vec{0})}$$

$$\Rightarrow \quad d\vec{x} = \frac{\partial}{\partial \vec{a}} \vec{f}_{(\vec{x},\vec{a})}|_{\vec{a}=\vec{0}} d\vec{a}$$

Let's put $\vec{t}_{(\vec{x})} = [\frac{\partial}{\partial \vec{a}} f_{(\vec{x},\vec{a})}]|_{\vec{a}=\vec{0}}$, so $d\vec{x} = \vec{t}_{(\vec{x})} d\vec{a}$
We can write it in components:

$$dx_i = [\frac{\partial}{\partial a^\mu} f_{i(\vec{x},\vec{a})}]|_{\vec{a}=\vec{0}} da^\mu = t_{i\mu} da^\mu \qquad i \in \hat{n}, \mu \in \hat{r}$$

Now we will discuss the change of function $F_{(\vec{x})}$ under the transformation $d\vec{a}$.

$$\begin{aligned} dF = \frac{\partial F_{(\vec{x})}}{\partial \vec{x}} d\vec{x} = \sum_i \frac{\partial F_{(\vec{x})}}{\partial x_i} dx_i = \sum_{i,\mu} \frac{\partial F_{(\vec{x})}}{\partial x_i} t_{i\mu(\vec{x})} da^\mu = \\ = \sum_{i,\mu} da^\mu \{ t_{i\mu(\vec{x})} \frac{\partial}{\partial x_i} \} F_{(\vec{x})} = -i \sum_\mu da^\mu \hat{L}_{\mu(\vec{x})} F_{(\vec{x})} \end{aligned}$$

Where the **r** quantities $\hat{L}_\mu = i \sum_i t_{i\mu(\vec{x})} \frac{\partial}{\partial x_i}$ are generators of the group. That's because the transformed quantity $F_{(\vec{x},\vec{a})}$ has to be obtained by successive transformations from $F_{(\vec{x})}$. So $da_\mu = \frac{a_\mu}{N}$ and

$$\begin{aligned} F_{(\vec{x},\vec{a})} = \lim_{N\rightarrow+\infty} (F_{(\vec{x},\vec{0})} + dF)^N &= \lim_{N\rightarrow+\infty} (F_{(\vec{x},\vec{0})} - i \sum_\mu da^\mu \hat{L}_{\mu(\vec{x})} F_{(\vec{x})})^N = \\ = \lim_{N\rightarrow+\infty} F_{(\vec{x},\vec{0})} (1 - i \sum_\mu da^\mu \hat{L}_{\mu(\vec{x})})^N &= \lim_{N\rightarrow+\infty} F_{(\vec{x},\vec{0})} (1 - i \sum_\mu \frac{a^\mu}{N} \hat{L}_{\mu(\vec{x})})^N = \\ = e^{-i \sum_\mu \hat{L}_\mu a^\mu} F_{(\vec{x},\vec{0})} &= \hat{U}_{(\vec{x},\vec{a})} F_{(\vec{x},\vec{0})} \end{aligned}$$

So $\hat{U}_{(\vec{x},\vec{a})} = e^{-i \sum_\mu \hat{L}_\mu a^\mu}$ are group operators and $\hat{L}_\mu$ are generators.

8.2 Special unitary group SU(N) and it's representation

A unitary operator represented by unitary matrix $n \times n$ can be written as

$$\hat{U} = e^{i\hat{H}}$$

where $\hat{H}$ is hermitian operator represented by matrix $n \times n$. All such operators form a group under matrix multiplication called $U_{(N)}$. Because $\hat{H}$ is hermitian, the diagonal matrix elements are real

$$\hat{H}^\dagger = \hat{H} \quad \Rightarrow \quad \hat{H}^*_{ii} = \hat{H}_{ii} \quad \forall i \in \hat{n}$$

It is obvious that $\hat{U}$ depends on n^2 real independent parameters. The group $U_{(N)}$ is continuously connected and represents a compact Lie group. For the unitary matrix holds that $|det\hat{U}| = 1$.

Theorem 34 *If $\hat{U} \in U_{(N)}$ than $det\hat{U} = e^{iTr\hat{H}}$*

Proof 4 *$\hat{H}$ is hermitian, so all diagonal elements are real and therefore the exponential on the right side is good defined. Matrix $\hat{U}$ is unitary and so we can diagonalize it. Let S be the matrix which describes the diagonalization so that $\hat{U}' = \hat{S}\hat{U}\hat{S}^{-1}$ and $\hat{U}'$ is diagonal.*

$$det\hat{U}' = det\begin{pmatrix} U'_{11} & & 0 \\ & \ddots & \\ 0 & & U'_{nn} \end{pmatrix} = det\hat{S}\hat{U}\hat{S}^{-1} = det\hat{S}det\hat{S}^{-1}det\hat{U} = det\hat{U}$$

If $\hat{U}'$ is diagonal than $\hat{H}$ has to be diagonal too.

$$\begin{aligned} det\hat{U} = det\hat{U}' = det\; e^{i\hat{H}'} &= det\; e^{i\begin{pmatrix} H'_{11} & & 0 \\ & \ddots & \\ 0 & & H'_{nn} \end{pmatrix}} \\ = det\begin{pmatrix} e^{iH'_{11}} & & 0 \\ & \ddots & \\ 0 & & e^{iH'_{nn}} \end{pmatrix} &= \prod_{i=1}^{n} e^{iH_{ii}} = e^{i\sum_{i=1}^{n} H_{ii}} \\ = e^{iTrH'} &= e^{iTrH} \end{aligned}$$

because $Tr\hat{H}' = Tr\hat{S}\hat{H}\hat{S}^{-1} = Tr\hat{H}\hat{S}\hat{S}^{-1} = Tr\hat{H}$ where we have used the equation $Tr\hat{A}\hat{B} = Tr\hat{B}\hat{A}$

Q.E.D

If we put another condition on group members

$$det\hat{U} = +1$$

there will be $n^2 - 1$ independent parameters. Such operators form a continuous compact Lie group denoted by $SU_{(N)}$. Obviously $SU_{(N)}$ is subgroup of $U_{(N)}$. If we take $\hat{U} \in U_{(N)}$ and $\hat{U}_0 \in SU_{(N)}$ the relation between these groups is manifested by relation

$$\hat{U} = \hat{U}_0(e^{i\frac{\alpha}{n}\mathbb{1}}) \qquad \hat{H} = \hat{H}_0 + \frac{\alpha}{n}\mathbb{1} \quad \wedge \quad \alpha = Tr\hat{H}$$

From this follows that every element of $U_{(N)}$ can be decomposed to an $SU_{(N)}$ element multiplied by $U_{(1)}$ element. Generators: The group $U_{(N)}$ has n^2 generators λ_j. From infinitesimal transformation

$$\hat{U}_{(\delta\Phi_j)} = e^{i\hat{H}_{(\delta\Phi_j)}} = \mathbb{1} - i\hat{H}_{(\delta\Phi_j)} = \mathbb{1} - i\sum_{i=1}^{n^2}\delta\Phi_j\hat{\lambda}_j$$

From hermicity of $\hat{H}$ we choose n^2 linearly independent hermitian matrixes as generators and consequent relation holds

$$[\hat{\lambda}_i, \hat{\lambda}_j] = ic_{ijk}\hat{\lambda}_k$$

The generators of $SU_{(N)}$, in analogy to those of $SU_{(N)}$ can be chosen as $n^2 - 1$ linearly independent hermitian matrixes with trace equal to zero! The same commutation relations holds for $SU_{(N)}$.

8.2.1 Special unitary group SU(2)

The $SU_{(2)}$ consists of two-dimensional unitary unimodular matrixes, which contains three parameters. As generators we need three linearly independent traceless matrixes. We choose

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

referred as Pauli matrixes. The commutation relations read $[\hat{\sigma}_i, \hat{\sigma}_j] = 2i\epsilon_{ijk}\hat{\sigma}_k$

We can simplify it by introducing $\hat{S}_j = \frac{1}{2}\hat{\sigma}_j$, which holds $[\hat{S}_i, \hat{S}_j] = i\epsilon_{ijk}\hat{S}_k$. Operators of $SU_{(2)}$ can be expressed as

$$\hat{U} = e^{-i\sum_{j=1}^{3}\Phi_j\hat{S}_j}$$

Maximum number of commutating generators is 1, so is the rank and from Racah theorem we have 1 Casimir operator

$$\hat{S}^2 = \hat{S}_1^2 + \hat{S}_2^2 + \hat{S}_3^2$$

So the $SU_{(2)}$ multiplets are characterized with eigenvalue of only one operator.

8.2.2 Special unitary group SU(3)

The group $SU_{(3)}$ has 8 generators denoted as $\hat{\lambda}_1, \hat{\lambda}_2, \dots, \hat{\lambda}_8$. Because $SU_{(2)}$ is subgroup of $SU_{(3)}$, we can construct $\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3$ from $\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3$ by extending them to three dimensions

$$\hat{\lambda}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \hat{\lambda}_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \hat{\lambda}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Others can be constructed from $\hat{\lambda}_1$ and $\hat{\lambda}_2$ by shifting non-zero elements

$$\hat{\lambda}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \rightarrow \quad \hat{\lambda}_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \rightarrow \quad \hat{\lambda}_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\hat{\lambda}_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \rightarrow \quad \hat{\lambda}_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \rightarrow \quad \hat{\lambda}_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

Finally $\hat{\lambda}_8$ is determined as

$$\hat{\lambda}_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

The factor $\frac{1}{\sqrt{3}}$ was chosen in order to $Tr\hat{\lambda}_i\hat{\lambda}_j = 2\delta_{ij} \quad \forall i, j \in \hat{8}$ hold. All $\hat{\lambda}'s$ are hermitian and traceless and $[\hat{\lambda}_i, \hat{\lambda}_j] = 2if_{ijk}\hat{\lambda}_k$ from a closed Lie algebra. The $\hat{\lambda}_3$ and $\hat{\lambda}_8$ are already diagonal. That implies rank of the group is 2 and there are 2 Casimir operators. So each multiplet will be characterized with eigenvalues of two Casimir operators. It is useful to redefine the generators as $\hat{F}_j = \frac{1}{2}\hat{\lambda}_j$ so that

$$[\hat{F}_i, \hat{F}_j] = if_{ijk}\hat{F}_k$$

holds. Now we will show the spherical representation of the $\hat{F}$ operators:

$$\begin{aligned} \hat{T}_\pm &:= \hat{F}_1 \pm i\hat{F}_2 \\ \hat{U}_\pm &:= \hat{F}_6 \pm i\hat{F}_7 \\ \hat{V}_\pm &:= \hat{F}_4 \pm i\hat{F}_5 \\ \hat{T}_3 &:= \hat{F}_3 \\ \hat{Y} &:= \frac{2}{\sqrt{3}}\hat{F}_8 \end{aligned} \tag{8.1}$$

It is simply just a transition from one group of generators to another. And commutation relations for this representation are

$$
\begin{array}{ll}
[\hat{T}_3, \hat{T}_\pm] = \pm \hat{T}_\pm \ , & [\hat{T}_+, \hat{T}_-] = 2\hat{T}_3 \\
[\hat{T}_3, \hat{U}_\pm] = \mp \frac{1}{2} \hat{U}_\pm \ , & [\hat{U}_+, \hat{U}_-] = \frac{3}{2}\hat{Y} - \hat{T}_3 =: 2\hat{U}_3 \\
[\hat{T}_3, \hat{V}_\pm] = \pm \frac{1}{2} \hat{V}_\pm \ , & [\hat{V}_+, \hat{V}_-] = \frac{3}{2}\hat{Y} + \hat{T}_3 =: 2\hat{V}_3 \\
[\hat{Y}, \hat{T}_\pm] = 0 \ , & [\hat{T}_3, \hat{Y}] = 0 \\
[\hat{Y}, \hat{U}_\pm] = \pm \hat{U}_\pm \ , & [\hat{Y}, \hat{V}_\pm] = \pm \hat{V}_\pm \\
[\hat{T}_+, \hat{V}_+] = & [\hat{T}_+, \hat{U}_-] = 0 \\
[\hat{U}_+, \hat{V}_+] = 0 \ , & [\hat{T}_+, \hat{V}_-] = -\hat{U}_- \\
[\hat{T}_+, \hat{U}_+] = \hat{V}_+ \ , & [\hat{U}_+, \hat{V}_-] = \hat{T}_-
\end{array}
$$

Furthermore $\hat{T}_+ = \hat{T}_-^\dagger \quad \hat{U}_+ = \hat{U}_-^\dagger \quad \hat{V}_+ = \hat{V}_-^\dagger$ and we can write Casimir operators as

$$
\begin{aligned}
\hat{C}_1 &= \sum \hat{F}_i^2 \\
\hat{C}_2 &= \sum d_{ijk} \hat{F}_i \hat{F}_j \hat{F}_k = \hat{C}_1 (2\hat{C}_1 - \frac{11}{6})
\end{aligned}
$$

8.2.3 Subalgebras of SU(3);shifting operators

In order to become familiar with $SU_{(3)}$ algebra, we study some of its subalgebras. Operators $\hat{T}_3, \hat{T}_+, \hat{T}_-$ form a closed subalgebra

$$[\hat{T}_+, \hat{T}_-] = 2\hat{T}_3 \ , \ [\hat{T}_3, \hat{T}_\pm] = \pm \hat{T}_\pm$$

The same holds for $\{\hat{U}_3, \hat{U}_-, \hat{U}_+\}$ and $\{\hat{V}_3, \hat{V}_-, \hat{V}_+\}$. We denote it T,V,U-spin algebras for all three of them are subalgebras of $SU_{(3)}$ and they are isomorphic to $SU_{(2)}$ the spin algebra. The operators $\hat{T}_\pm, \hat{U}_\pm, \hat{V}_\pm$ are also shift operators. From equation $[\hat{Y}, \hat{T}_3] = 0$ comes that operators $\hat{Y}, \hat{T}_3$ may be simultaneously diagonalized so we will show effect of these operators on an eigenstate $|Y, T_3>$. Operators $\hat{V}_\pm$ transform a state with quantum numbers T_3 and Y into a state $T_3 \pm \frac{1}{2}$ and $Y \pm 1$. Operators $\hat{U}_\pm$ transform a state with quantum numbers T_3 and Y into a state $T_3 \mp \frac{1}{2}$ and $Y \pm 1$. Operators $\hat{T}_\pm$ transform a state with quantum numbers T_3 and Y into a state $T_3 \mp 1$ and Y.

Let's review some facts about T,U,V spin algebras and their coupling:

1. $SU_{(3)}$ algebra has T,U,V subalgebras;each isomorphic t o $SU_{(2)}$ Therefore $SU_{(3)}$ multiplets can be constructed by means of coupled T,U,V multiplets.

2. The operators $\hat{T}_3, \hat{Y}$ and also $\hat{U}_3 = \frac{1}{2}(\frac{3}{2}\hat{Y} - \hat{T}_3)$ and $\hat{V}_3 = \frac{1}{2}(\frac{3}{2}\hat{Y} + \hat{T}_3)$ can be simultaneously diagonalized with eigenvalues

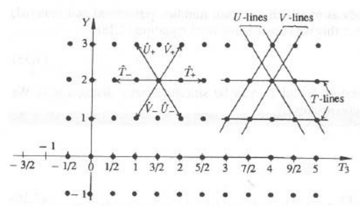

Figure 8.1: The effect of shift operators in Y - T_3 plane

$$T_3\ ,\ Y\ ,\ \frac{1}{2}(\frac{3}{2}Y - T_3)\ ,\ \frac{1}{2}(\frac{3}{2}Y + T_3)$$

3. The shift operators act on the states of $SU_{(3)}$ multiplet according to figure (ten nad tim). The end points of these operators are situated on a regular hexagon.

4. $SU_{(3)}$ multiplet is constructed from a T multiplet, V multiplet, U multiplet. These submultiplets must be coupled because of commutation relations.

5. From the equivalence of T,U,V the representations of $SU_{(3)}$ multiplets within the $Y - T_3$ plane have to be regular (not necessarily equilateral) hexagons or triangles.

6. From the same equivalence comes that a figure representing an $SU_{(3)}$ multiplet has to be symmetric with respect to the axis $T_3 = 0, U_3 = 0, V_3 = 0$

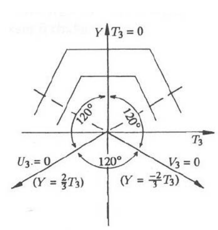

Figure 8.2: The axes orientation in Y - T_3 plane

7. The origin $Y = 0, T_3 = 0$ is the center of each multiplet.

Let's look at quantitative analysis of multiplets. Consider the state which belongs to the largest T_3 value in the mutiplet("the maximum weight state")

$$\psi_{max} = |T_{3max}\psi >$$

and $\hat{T}_+\psi_{max} = \hat{V}_+\psi_{max} = \hat{U}_-\psi_{max} = 0$ otherwise it would raise the T_3 value. The boundary of multiplet can be constructed by repeated application of $\hat{V}_-$ on ψ_{max}. Let's assume it can be done p times and then

$$\hat{V}_-^{p+1}\psi_{max} = 0$$

As soon as $\hat{V}_-^{p}\psi_{max}$ is reached we may follow the boundary of the multiplet by a repeated action of $\hat{T}_-$ on a state. Let's say it can be done q times till

$$\hat{T}_-^{q+1}\hat{V}_-^{p+1}\psi_{max} = 0$$

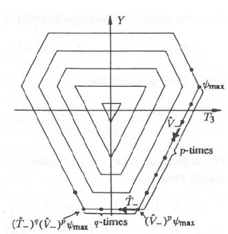

Figure 8.3: The boundary in the SU(3) multiplet

The numbers p and q define a multiplet of the group $SU_{(3)}$. We have done this because it can be proven that the boundary of the multiplet always has to be convex. The mesh points of the boundary of an $SU_{(3)}$ multiplet are occupied by only one state. On the next layer of weight diagram each mesh point is occupied by two states. The following shell has triple occupancy and so on till the hexagon changes into the triangle (let's say after r steps). Now every shell point carries r+1 states. From now on each point has has the same r+1 multiplicity. It is described in figure 8.4.

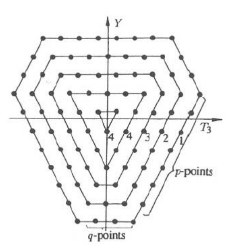

Figure 8.4: The multiplicity of states in the shell

Or it can be said that $\hat{T}_-$ increases the number of states each time it jumps from one shell to another till it comes through zero.

8.3 Physical connection to multiplets

We have already shown that $SU_{(3)}$ symmetry yields to multiplet structure. Question remains how to interpret it in the reality. In searching for physical interpretation of $SU_{(3)}$the crucial step is to understand consequences of the $SU_{(3)}$ representation and its quantum numbers T_3 and Y. The T,U,V spin fulfill the angular momentum algebra and forms subalgebras of $SU_{(3)}$. That will enable us to classify elementary particles within $SU_{(3)}$ multiplets if we interpret Y as hypercharge and T as isospin. We know that isospin multiplets in a given $SU_{(3)}$ multiplet are given by parallels to the T_3 axis. In the first step we define charge operator by

$$\hat{Q} = \frac{1}{2}\hat{Y} + \hat{T}_3$$

We denote the $SU_{(3)}$ states by $|T_3 Y \alpha >$, where α abbreviates additional quantum numbers given by two Casimir operators which classify the multiplets uniquely. In other words α specifies which multiplet we take and $T_3 Y$ defines the position in multiplet. The eigenvalue equations

$$\begin{aligned} \hat{Y}|T_3 Y \alpha > &= Y|T_3 Y \alpha > \\ \hat{T}_3|T_3 Y \alpha > &= T_3|T_3 Y \alpha > \end{aligned}$$

gives for charge operator

$$\hat{Q}|T_3 Y \alpha >= \frac{Y}{2} + T_3|T_3 Y \alpha >$$

Let us now look at the smallest non trivial representation of $SU_{(3)}$. As we know, the isospin doublet with $T = \frac{1}{2}$ is the smallest nontrivial representation of isospin $SU_{(2)}$. This implies that we can construct all higher multiplets from this representation. Technically it is achieved by Clebsch-Gordan coupling of isospins $T = \frac{1}{2}$ to arbitrary isospin. This cannot be performed with the lowest $SU_{(2)}$ multiplet $T = 0$. In that sense the $T = 0$ multiplet of $SU_{(2)}$ is trivial. Because the $SU_{(3)}$ F-spin algebra contains the isospin as a subalgebra, the smallest $SU_{(3)}$ representation we are looking for must contain at least one $T = \frac{1}{2}$ charge doublet. Obviously T-spin, U-spin and V-spin algebras are fully symmetric in F-spin algebra. Consequently, the $SU_{(3)}$ multiplet must contain T,U,V doublets. So we are led to two equilateral triangles in $Y - T_3$ plane. As required by symmetries, they are centered around the origin. We denote them [3] and $[\bar{3}]$ because they contain 3 states. See figure 8.5

If [3] stands for particles, then $[\bar{3}]$ represents the corresponding antiparticles since the state $\bar{\psi}_\nu; \nu \in \hat{3}$ has opposite hypercharge and T_3 and thus the charge compared to ψ_ν.

$$\hat{Q}\psi_\nu = Q_\nu \psi_\nu \qquad \hat{Q}\bar{\psi}_\nu = -Q_\nu \bar{\psi}_\nu$$

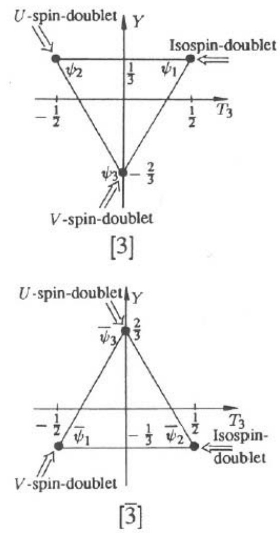

Figure 8.5: The smallest nontrivial representations $SU_{(3)}$

Each of these two representations [3] and [$\bar{3}$] contains an isodoublet $T = \frac{1}{2}$ and isosinglet $T = 0$.

$$\psi_1 = |\frac{1}{2}Y> \quad \psi_2 = |-\frac{1}{2}Y> \quad \psi_3 = |0Y>$$

Now we can determine the hypercharge of each state because we know isospin of each state

$$\hat{T}_3\psi_1 = \frac{1}{2}\psi_1 \quad \hat{T}_3\psi_2 = -\frac{1}{2}\psi_2 \quad \hat{T}_3\psi_3 = 0\psi_3.$$

We can see from figure 8.5 that ψ_1 is U-spin singlet. Therefore

$$\hat{U}_3\psi_1 = 0$$

From definitions of F-spin algebra we can write

$$\hat{U}_3 = \frac{3\hat{Y} - 2T_3}{4}$$

and therefore

$$\begin{aligned} \hat{Y}\psi_1 &= \tfrac{1}{3}\psi_1 & \hat{Y}\bar{\psi}_1 &= -\frac{1}{3}\bar{\psi}_1 \\ \hat{Y}\psi_2 &= \tfrac{1}{3}\psi_2 & \hat{Y}\bar{\psi}_2 &= -\frac{1}{3}\bar{\psi}_2 \\ \hat{Y}\psi_3 &= -\tfrac{2}{3}\psi_3 & \hat{Y}\bar{\psi}_3 &= \frac{2}{3}\bar{\psi}_3 \end{aligned}$$

This leads to charge eigenvalues

$$\begin{aligned} \hat{Q}\psi_1 &= \tfrac{2}{3}\psi_1 & \hat{Q}\bar{\psi}_1 &= -\frac{2}{3}\bar{\psi}_1 \\ \hat{Q}\psi_2 &= -\tfrac{1}{3}\psi_2 & \hat{Q}\bar{\psi}_2 &= \frac{1}{3}\bar{\psi}_2 \\ \hat{Q}\psi_3 &= -\tfrac{1}{3}\psi_3 & \hat{Q}\bar{\psi}_3 &= \frac{1}{3}\bar{\psi}_3 \end{aligned}$$

This has far-reaching consequences because it presents the state with fractional charge. We can identify states ψ_ν with quarks q_ν and $\bar{\psi}_\nu$ with antiquarks $\bar{q}_\nu$. Many physicists have searched for free quarks (L.W.Jones, C.B.A McCusher, I. Cairus, W.M. Fairbank) but it seems there is a physical law which forbids the unbound existence of quarks. By applying F-spin operators to quark states we can prove that transformation properties lead to unitary operators with $det\hat{U} = 1$. Furthermore

$$|q_\nu>' = \hat{U}_{(\vec{\Theta})}|q_\nu> = \sum_\nu |q_\mu > U_{\mu\nu}(\vec{\Theta})$$

$$U_{\mu\nu}(\vec{\Theta}) = < q_\mu|U_{(\vec{\Theta})}|q_\nu > \qquad U_{(\vec{\Theta})} = e^{-i\sum_\alpha \Theta_\alpha \hat{F}_\alpha}$$

Generators λ_α of $SU_{(3)}$ in the [3] representation are equal to Gell-Mann matrixes. Transformation properties of $[\bar{3}]$ representation are the same as for [3] when we introduce

$$\hat{\bar{\vec{F}}} = -\hat{\vec{F}}^*$$

and

$$U'_{(\vec{\Theta})} = U^*_{(\vec{\Theta})} = e^{-i\vec{\Theta}\hat{\bar{\vec{F}}}}$$

Note that $-\hat{F}_i^* \rightarrow \hat{\bar{F}}_i$ has nothing to do with hermitian conjugation. In fact there does not exist any unitary transformation connecting $\hat{U}$ and $\hat{U}'$ and therefore [3] and $[\bar{3}]$ are independent representations. If the representation were to be equivalent, their generators would only differ by a unitary transformation $\hat{S}$

$$\hat{S}\hat{F}_\alpha\hat{S}^{-1} = \hat{\bar{F}}_\alpha \quad \Leftrightarrow \hat{S}\hat{\lambda}_\alpha\hat{S}^{-1} = \hat{\bar{\lambda}}_\alpha$$

If λ is the eigenvalue of $\hat{\lambda}_\alpha$ then

$$\hat{S}\hat{\lambda}_\alpha|q_i> = \hat{S}\lambda|q_i> = \hat{S}\lambda\hat{S}^{-1}\hat{S}|q_i>$$

Now we abbreviate $\hat{S}|q_i> =: |q_i>'$ and therefore also $-\hat{\lambda}_\alpha^*|q_i>' = \lambda|q_i>'$ and both $\hat{\lambda}_\alpha$ and $-\hat{\lambda}_\alpha^*$ has the same eigenvalue. The $\hat{\lambda}_\alpha$ are hermitian, so $\hat{\lambda}_\alpha = \hat{\lambda}_\alpha^\dagger = (\hat{\lambda}_\alpha^*)^T$ and

$$det(\hat{\lambda}_\alpha - \lambda\hat{\mathtt{1}}) = det(\hat{\lambda}^*_\alpha - \lambda\hat{\mathtt{1}}) = 0$$

If we calculate this equation for all α we will come to contradiction for $det(\hat{\lambda}_8)$. Therefore both representations are independent. Now we have to satisfy that all $SU_{(3)}$ multiplets can be composed from [3] and $[\bar{3}]$. This is done by means of representation coupling. In principal the construction requires only one of two fundamental representations [3] or $[\bar{3}]$, because one of them can be derived via Kronecker product

$$[3] \otimes [3] = [6] \oplus [\bar{3}]$$
$$[\bar{3}] \otimes [\bar{3}] = [\bar{6}] \oplus [3]$$

However for physical reasons we need both of them because quarks and antiquarks differ by their baryon number and charge. The general Kronecker product of $SU_{(3)}$ representation contains p triplets and q antitriplets.

$$[3] \otimes [3] \otimes [3] \otimes [3] \otimes \ldots \otimes [\bar{3}] \otimes [\bar{3}] \otimes [\bar{3}] \otimes \ldots$$

The (p,q) state of maximum weight is that one which consists of p quarks of maximal weight and q quarks of maximal weight (quark states $|\frac{1}{2}, \frac{1}{3} >$ and antiquark states $|\frac{1}{2}, -\frac{1}{3} >$). This state is characterized by

$$(T_3)_{max} = \frac{p+q}{2} \qquad (Y)_{max} = \frac{p-q}{3}$$

From this state we can generate the whole multiplet by means of the shift operators. We can derive that largest multiplet of p quark q antiquark configuration represents the D(p,q) multiplet. This is the most important connection between group theory and quark hypothesis.

Here we present simplest multiplets of $SU_{(3)}$. Obviously the most interesting ones are D(3,0) for baryons and D(1,1) for mesons. At the end lets say some rules for the decomposition of Kronecker product

$$N \otimes N = \frac{N-1}{2} \oplus \frac{N+1}{2} \qquad N \otimes \bar{N} = 1 \oplus N^2 - 1$$

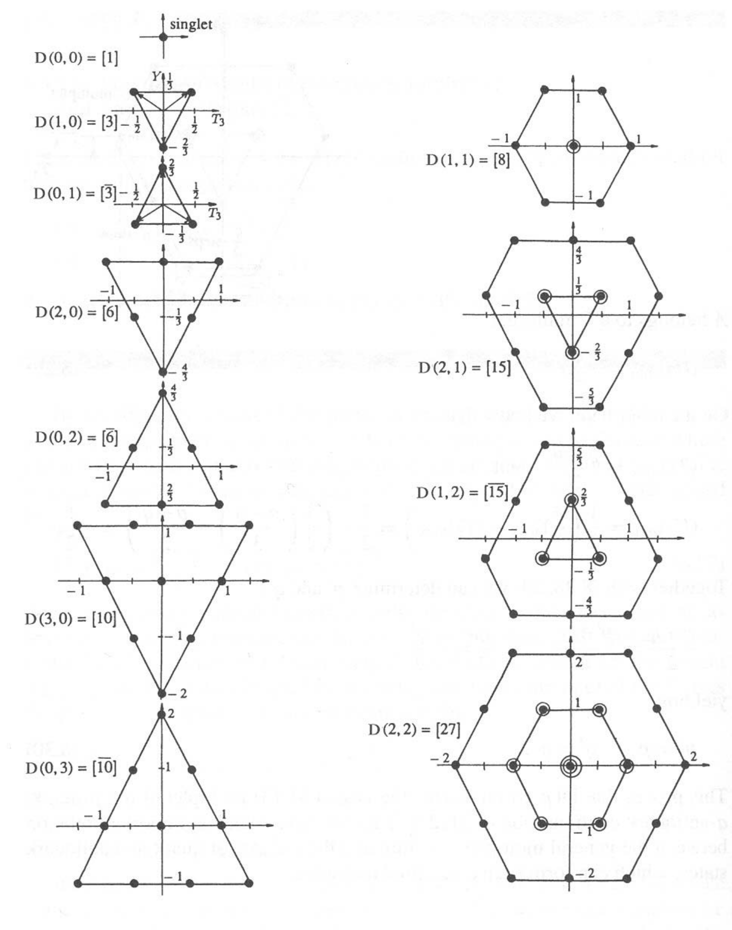

Figure 8.6: The simplest multiplets of $SU_{(3)}$

Chapter 9

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