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Axiomatics of Quantum Theory and Operator Algebra

Diploma thesis

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Název práce:	**Axiomatika kvantové teorie a operátorové algebry**
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Abstrakt:	Nalezneme dolní hranici pro disperze stavů na Jordanově algebře. Vyšetříme maximální porušení Bellové nerovnosti pro JBW algebru. Zobecníme Bellovu nerovnost pro obecnější korelační dualitu.
Klíčová slova:	Operátorová algebra, disperze stavů, Bellovy nerovnosti, korelační dualita

Title:	**Axiomatics of Quantum Theory and Operator Algebra**
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Abstract:	We find lower bound for overall disperion on Jordan algebra. We investigate maximal violation of Bell's inequality on JBW algebra. We generalize Bell's inequality for more general correlation duality.
Key words:	Operator algebra, dispersion of states, Bell's inequalities, correlation duality

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Chapter 1

Introduction

The Copenhagen interpretation of quantum mechanics is one formulated by Niels Bohr and Werner Heisenberg around 1927. Viewing the nature as inherently random it makes only (in contrast to Newtonian physics) statistical predictions about experiments. Quantum mechanics only assigns probabilities to each of a range of possible outcomes while in compliance with Heisenberg's uncertainty principle it is impossible to make predictions with arbitrarily better accuracy. That is, we can only say, for example, where particle is at the cost of being able to say how fast it is moving. At a fundamental level, randomness is thus indiscerptibly embedded in the reality.

The question is whether properties of individual systems possess values prior to the measurement that reveals them; not whether there are laws enabling us to predict at an earlier time what those values will be. Uncertainty principle simply tells us that we cannot know definite position and velocity of an electron at any instant of time. It does not tell us that electron, at any instant of time, does not possess definite position and velocity. It is no surprise that the idea of nature being fundamentally random was met with some resistance, giving rise to unceasing debates about possibility of building a deeper formalism which would make it possible to attain definite value to both particle's position and velocity. Such theory would be to quantum mechanics as classical mechanics to classical statistical mechanics.

General feature of hidden-variables is as follows: given an ansemble of identical physical systems all prepared in the state φ described by observables $A, B, C, \ldots$ such a theory should assign to each individual member of that ensemble a set of numerical values for each observable, $v(A), v(B), v(C), \ldots$. The theory should provide a rule for every state φ telling us how to distribute those values over the members of the ensemble described by φ in such a way that the statistical distribution of outcomes, for any measurement quantum mechanics permits, agrees with the predictions of quantum mechanics.

First attempt to show that hidden variables theory of quantum physics was impossible was made in 1932 by John von Neumann which later turned out to be deeply flawed. Von Neumann employed one of the constraints of quantum mechanics which requires that if $A, B, C, \ldots$ is a mutually commuting subset of observables then any functional identity

$$f(A, B, C, \ldots) = 0$$

holds for their simultaneous eigenvalues also, i.e.

$$f(v(A), v(B), v(C), \ldots) = 0\,.$$

A particular consequence of the above equations is that if A and B commute then $f(A, B, C) = A + B - C = 0$ must satisfy $f(v(A), v(B), v(C)) = v(A) + v(B) - v(C) = 0$. However, von Neumann imposed this condition on a hidden variables theory even if A and B do not commute which (according to John Bell) was *silly*. When A and B do not commute they do not have simultaneous eigenvalues nor they can be simultaneously measured. Von Neumann was led to it because it holds in the mean: for any state φ, quantum mechanics requires, whether or not A and B commute, that

$$< \varphi|A + B|\varphi > = < \varphi|A|\varphi > + < \varphi|B|\varphi > \ .$$

It is easy to see that the results of quantum mechanics are incompatible with values satisfying this condition: consider two-dimensional state space describing a single spin $\frac{1}{2}$. Let $A = \sigma_x$, $B = \sigma_y$. The values $v(A)$ and $v(B)$ are each restricted to ± 1, thus the only values $v(A)+v(B)$ can have are $-2, 0$, and 2. This is in a contradiction with $v(A + B) = \pm\sqrt{2}$. Therefore a hidden variables theory of this simple system cannot satisfy $v(A+B) = v(A)+v(B)$.

In 1935 Grete Hermann, a german mathematician, refuted von Neumann's proof in her paper *Die Naturphilosophischen Grundlagen der Quantenmechanik* which unfortunately went unnoticed. Later that year, on May 15, Albert Einstein together with his two students Boris Podolsky and Nathan Rosen presented what is often called the Einstein-Podolsky-Rosen (EPR) Paradox in their paper *Can Quantum-Mechanical Description of Reality be Considered Complete?* Aiming towards the effort to show the existence of hidden variables the authors attacked various concepts of quantum mechanics concluding that quantum theory was not complete yet. They believed that the shadow of doubt cast upon quantum mechanics could be cleared up by incorporating hidden variables. On October 15 that year Niels Bohr retorted with his paper explaining that a viewpoint termed *complementarity* would seem to fulfill, within its scope, all rational demands of completeness. In this manner Bohr whisked away *ambiguous* formulation of Einstein, Podolsky and Rosen. Another paper defying all impossibility proofs was published in 1952 by David Bohm.

In 1957 Andrew Gleason published theorem which states that the only possible probability measures μ on Hilbert spaces of dimension at least three are of the form $\mu(X) = Tr(\rho P_X)$, where ρ is a positive semi-definite self-adjoint operator of unit trace, and where P_X is a projection operator for projection onto the subspace X. The original proof of the theorem however acquired reputation of being hard to grasp. Attempts to make the proof more transparent lead to geometrical lemmas that possess also easy proofs of some consequences of Gleason theorem.

John Stewart Bell gave arguments that the prevailing view that these results disprove hidden variables is actually a false one. According to Bell what these theorems show is that hidden variables must allow for two important and fundamental quantum features: *contextuality* and *nonlocality.* To take contextuality into account, one must consider the results of a measurement to depend on the attributes of both the system and the measuring apparatus. The concept of contextuality lies at the heart of both Gleason's theorem and the theorem od Kochen and Specker. However, its full meaning and importance is not exposed in the proofs nor discussed. The exception to this is two works by Bell in which term contextuality is made quite clear. The concept of nonlocality is well known through Bell's famous mathematical theorem in which he addresses the problem of EPR paradox. In a system exhibiting nonlocality the consequences of events at one place propagate to other places instantaneously. Although Einstein, Podolsky and Rosen were attempting to demonstrate a different conclusion (the incompleteness of the quantum theory), their analysis serves to point out the conditions under which (as would become evident after Bell's work) nonlocality arises. Bell states that nonlocality is deeply rooted in quantum mechanics itself and will persist in any completion concluding that quantum theory is nonlocal.

The aim of this work is to investigate dispersions of states on Jordan algebras and generalization of Bell's inequalities for more general correlation duality.

In the chapter 2 we briefly summarize important results in the structure theory of operator algebras, namely C^*, von Neumann and Jordan algebras. Chapter 3 deals with quantum history approach.

Chapter 4 deals with Hidden variables theory. In section 4.3 we show that Hidden variables theory is excluded in Jordan algebras (see Theorem (43)).

In chapter 5 we summarize basic results about so-called Bell inequalities. In the section 5.3 we show that in certain type of JBW algebra we can find a state maximally violating Bell's inequality (see Theorem (51)). Next we generalize Bell's inequalities for more general correlation duality (see Theorem (53)) and give an example of saturation.

Chapter 2

Operator algebras

The principal mathematical tools are the algebras of bounded operators $\mathcal{B}(\mathcal{H})$ on a separable Hilbert space $\mathcal{H}$. There are three main mathematical structures used to build the framework of quantum and algebraic quantum field theory - namely C^*, W^* and Jordan algebras.

2.1 C^*-algebras

An associative algebra $(\mathcal{A}, +, \cdot, *)$ is called a *$*$-algebra* if the operation $*$ obeys the following rules for all $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{C}$: (i) $a^{**} = a$ (ii) $(a+b)^* = a^* + b^*$ (iii) $(\lambda a)^* = \overline{\lambda} a^*$ (iv) $(ab)^* = b^* a^*$.

Such algebra endowed with the norm $|| \cdot ||$ is a Banach space. $\mathcal{A}$ is said to be an *involutive Banach algebra* if the following conditions are satisfied for all $a, b \in \mathcal{A}$: (i) $||ab|| \leq ||a|| \cdot ||b||$ (ii) $||a^*|| = ||a||$. A C^*-algebra is an involutive Banach algebra satisfying (iii) $||a^* a|| = ||a||^2$ for all $a \in \mathcal{A}$. The *centre*, $Z(\mathcal{A})$, of a C^*-algebra $\mathcal{A}$ is the set of all elements commuting with every element in $\mathcal{A}$.

Let $M_n(\mathbb{C})$ be the complex algebra of $n \times n$ complex matrices endowed with the standard arithmetic operations and with the involution sending a matrix $a \in M_n(\mathbb{C})$ to its adjoint matrix a^*. Then $M_n(\mathbb{C})$ with the matrix norm is a finite-dimensional C^*-algebra.

Let X be a locally compact Hausdorff space. Let us denote by $C_0(X)$ the $*$-algebra of all continuous functions on X such that for each $\epsilon > 0$ the set $\{x \in X \,|\, |f(x)| \geq \epsilon\}$ is compact and the $*$-operation assigns to each such function its complex conjugate. Let $||f|| := sup_{x \in X} |f(x)|$ be the norm on $C_0(X)$. Then $C_0(X)$ becomes an abelian (=commutative) C^*-algebra. If X is a compact space then $C_0(X)$ coincides with the algebra $C(X)$ of all continuous complex functions on X.

Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the $*$-algebra of all bounded operators on $\mathcal{H}$ with usual arithmetic operations and the $*$ operation that assigns to each operator its adjoint operator. Then $\mathcal{B}(\mathcal{H})$ endowed with the norm $||a|| := sup_{[\xi\in\mathcal{H},||\xi||=1]}||a\xi||$ is a C^*-algebra. Any norm-closed $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ is also a C^*-algebra.

Let F be a subset of $\mathcal{B}(\mathcal{H})$ that is closed under forming adjoint operators. Let us denote by F' the *commutant* of F, i.e. the set $F' := \{a \in \mathcal{B}(\mathcal{H}) \,|\, ax = xa \text{ for all } x \in F\}$. Then F' is a subalgebra of $\mathcal{B}(\mathcal{H})$.

Let $\mathcal{A}$ be abelian C^*-algebra. A *character* ω on $\mathcal{A}$ is a nonzero linear map $\omega : \mathcal{A} \mapsto \mathbb{C}$ for which $\omega(ab) = \omega(a)\omega(b)$ and $\omega(a^*) = \overline{(\omega(a))}$ for all $a, b \in \mathcal{A}$. The *spectrum*, $\Omega(\mathcal{A})$, of an algebra $\mathcal{A}$ is the set of all characters on $\mathcal{A}$ endowed with the topology of pointwise convergence on elements of $\mathcal{A}$. [1] The *Gelfand transform* is the map $\tau : \mathcal{A} \mapsto C_0\big(\Omega(\mathcal{A})\big)$ defined for all $a \in \mathcal{A}$ by the formula $\tau(a)(\omega) = \omega(a)$ for all $\omega \in \Omega(\mathcal{A})$.

Theorem 1 ***(Gelfand).*** *For each abelian C^*-algebra $\mathcal{A}$ the Gelfand transform is a $*$-preserving isometric isomorphism of $\mathcal{A}$ onto $C_0\big(\Omega(\mathcal{A})\big)$.*

As a result of the Gelfand theorem we can identify abelian C^*-algebra with algebra of continuous functions on locally compact spaces.

If a C^*-algebra, $\mathcal{A}$, contains a unit with respect to multiplication, $\mathbb{I}$, then $\mathcal{A}$ is said to be *unital.* In the rest of this paragraph $\mathcal{A}$ will always denote a C^*-algebra unless otherwise stated.

Let $x \in \mathcal{A}$. The *spectrum* of x is the set $\mathrm{Spec}(x) := \{\lambda \in \mathbb{C} \,|\, (x - \lambda\mathbb{I}) \text{ is not invertible in } \mathcal{A}\}$. We say that element $x \in \mathcal{A}$ is *normal* if $x^*x = xx^*$. The smallest C^*-algebra containing an element x will be denoted by $C^*(x)$. It is the norm closure of the set of all finite sums of the form $\sum_{i,j} a_{ij}x^i x^{*j}$, $a_{ij} \in \mathbb{C}$.

If x is normal element then $C^*(x)$ is abelian. We can apply the Gelfand theorem and identify $C^*(x)$ with the continuous functions on its spectrum $X = \Omega\big(C^*(x)\big)$. It turns out that X can be identified with $\mathrm{Spec}(x)\backslash\{0\}$ as the map $\omega \in X \mapsto \omega(x) \in \mathrm{Spec}(X\backslash\{0\})$ establishes a homomorphism between X and $\mathrm{Spec}(x)\,\backslash\{0\}$. In this manner the algebra $C^*(x)$ can be identified with the algebra $C_0(\mathrm{Spec}(x)\,\backslash\{0\})$.

An element $x \in \mathcal{A}$ is called *self-adjoint* if $x = x^*$. A self-adjoint part of $\mathcal{A}$ shall be denoted by $\mathcal{A}_{sa}$. A general element $x \in \mathcal{A}$ can be written as $x = y + iz$, where $y = \frac{x+x^*}{2} \in \mathcal{A}_{sa}$ and $z = -\frac{i(x-x^*)}{2} \in \mathcal{A}_{sa}$. The elements y and z are called the *real* and the *imaginary part* of x, respectively. An element of C^*-algebra is called *positive* (or *nonnegative*) if $x = a^*a$ for some $a \in \mathcal{A}$. An *approximate unit* is an upwards directed system $(a_\alpha)_{\alpha\in I}$ of nonnegative elements of $\mathcal{A}$ such that $lim_{\alpha\in I}||x - xu_\alpha|| = lim_{\alpha\in I}||x - u_\alpha x|| = 0$, for each

[1]It can be proved that $\Omega(\mathcal{A})$ is a locally compact Hausdorf space.

$x \in \mathcal{A}$.

An element $u \in \mathcal{A}$ is said to be *unitary* if $u^*u = uu^* = \mathbb{I}$. The spectrum of any unitary element is contained in the unit circle in the complex plane. If $\mathcal{A} = C(X)$ then the unitary elements are just continuous mappings of X onto the unit circle. If $\mathcal{A} = M_n(\mathbb{C})$, then the set of unitary elements is the set of unitary (orthonormal) matrices. The set of all unitary elements in $\mathcal{A}$ endowed with the multiplication forms the group, $U(\mathcal{A})$, called the *unitary group of* $\mathcal{A}$. As for any self-adjoint element x with $||x|| \leq 1$ we find that the elements $u = x + i(1-x^2)^{1/2}$ and $u^* = x - i(1-x^2)^{1/2}$ are unitary with $x = \frac{1}{2}(u + u^*)$, we see that $U(\mathcal{A})$ linearly generates the whole of $\mathcal{A}$. An element $s \in \mathcal{A}$ is called *symmetry* if it is unitary and self-adjoint.

A *projection* $p \in \mathcal{A}$ is a self-adjoint idempotent, i.e. $p = p^* = p^2$. In the algebra $C_0(X)$ the projections are exactly the characteristic functions of clopen subsets of X.

Any projection in the algebra $M_2(\mathbb{C})$ is of the form

$$\left(\begin{array}{cc} a & \sqrt{a-a^2}e^{i\varphi} \\ \sqrt{a-a^2}e^{-i\varphi} & a \end{array} \right),$$

where $a \in [0,1]$ and $\varphi \in \mathbb{R}$.

The set of all projections in $\mathcal{A}$ will be denoted by $P(\mathcal{A})$. If $\mathcal{A}$ is unital then the map $p \mapsto 2p - \mathbb{I}$ is a one-to-one correspondence between the projection structure and the set of all symmetries in $\mathcal{A}$. An element $v \in \mathcal{A}$ is called a *partial isometry* if $v^*v = p$, where $p \in P(\mathcal{A})$. In this case $vv^* = q$, where q is another projection in $P(\mathcal{A})$. The projections p and q are said to be the *initial* and the *final projection* of v, respectively.

A C^*-subalgebra B of a C^*-algebra $\mathcal{A}$ is defined as the norm closed $*$-subalgebra of $\mathcal{A}$. A C^*-subalgebra generated by elements $a_1, \ldots, a_n \in \mathcal{A}$ will be denoted by $C^*(a_1, \ldots, a_n)$. A closed subset $\mathcal{I}$ of $\mathcal{A}$ is called *closed left* (*right*) *ideal* if $ax \in \mathcal{I}$ ($xa \in \mathcal{I}$) for each $a \in \mathcal{A}$ and $x \in \mathcal{I}$. A closed subset in $\mathcal{A}$ is called *closed* (*two-sided*) *ideal* if it is simultaneously left and right ideal. The matrix algebra has no nontrivial ideals. The C^*-algebra having no nontrivial ideals is called *simple.*

A linear map $\pi : \mathcal{A} \mapsto B$ between C^*-algebras $\mathcal{A}$ and B is called a $*$-homomorphism if it preserves product and $*$-operation. Any $*$-homomorphism is norm-decreasing, and thereby continuous. The image $\pi(\mathcal{A})$ is a C^*-subalgebra of B. The kernel $Ker\,\pi = \{a \in \mathcal{A} \,|\, \pi(a) = 0\}$ is a closed ideal in $\mathcal{A}$, and $\pi(\mathcal{A})$ is isomorphic to $\mathcal{A}/Ker\,\pi$. An injective $*$-homomorphism of $\mathcal{A}$ onto B is called $*$-homomorphism. Any $*$-isomorphism is an isometry. A $*$-homomorphism of $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$ is also called $*$-representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$. If $Ker\,\pi = \{0\}$ then π is called *faithful.*

A linear form $f : \mathcal{A} \mapsto \mathbb{C}$ on a C^*-algebra $\mathcal{A}$ is called a *positive functional* if $f(a) \geq 0$ whenever $a \geq 0$. Any positive functional on some C^*-algebra is bounded. A positive functional ρ on a C^*-algebra is called *state* if $||\rho|| = 1$.

Very powerful tool is the so-called *Gelfand-Naimark-Segal construction* (GNS):

Theorem 2 *For any positive functional ρ on C^*-algebra $\mathcal{A}$ there is a Hilbert space, $\mathcal{H}_\rho$, a $*$-morphism, $\pi_\rho : \mathcal{A} \mapsto \mathcal{B}(\mathcal{H}_\rho)$, and a cyclic vector, $\xi_\rho \in \mathcal{H}_\rho$, such that*

$$\rho(x) = (\pi_\rho(x)\xi_\rho, \xi_\rho).$$

Moreover, the triple $(\pi_\rho, H_\rho, \xi_\rho)$ is unique up to unitary transform between the corresponding Hilbert spaces.

A state ρ on a C^*-algebra $\mathcal{A}$ is called *pure* if it cannot be written as a nontrivial convex combination of states, i.e. if the following implication holds: if $\rho = \frac{1}{2}(\rho_1 + \rho_2)$ for states ρ_1 and ρ_2 on $\mathcal{A}$, then $\rho_1 = \rho_2 = \rho$. If x is a nonzero positive element in a C^*-algebra $\mathcal{A}$, then there is a pure state ρ such that $\rho(x) = ||x||$.

The structure of the set of states in the dual of C^*-algebra is convex. A *face* F in a subset of vector space K is a convex subset of K such that the following implication holds: if $\alpha x + (1-\alpha)y \in F$ for $x, y \in K$ and $\alpha \in [0,1]$, then $x, y \in F$. An element $x \in K$ is called an *extreme point* of the set K if the set $F = \{x\}$ is a face of K.

Let $\mathcal{A}$ be a C^*-algebra. By the symbol $S(\mathcal{A})$ we shall denote the convex set of all states on $\mathcal{A}$. $S(\mathcal{A})$ is called the *state space of* $\mathcal{A}$. The extreme points of this set are pure states. A subset S of the state space $S(\mathcal{A})$ is called *order determining* if, and only if, $\rho(a) \geq 0$ for all $\rho \in S$ implies that $a \geq 0$.

Very significant role in the operator algebra theory is played by tensor products of C^*-algebras. Let X and Y be linear spaces. Let us denote by $X \otimes_{alg} Y$ their algebraic tensor product. It is linearly spanned by the simple tensors $x \otimes y$, $x \in X$, $y \in Y$. For each bilinear form $b : X \times Y \mapsto \mathbb{C}$ there is a unique linear form f on $X \otimes_{alg} Y$ such that $b(x, y) = f(x \otimes y)$, for all $x \in X$ and $y \in Y$. If B_1 and B_2 is a linear basis of X and Y respectively, then the set $\{b_1 \otimes b_2 \,|\, b_1 \in B_1,\, b_2 \in B_2\}$ is a linear basis of $X \otimes_{alg} Y$.

Let H and K be Hilbert spaces. Then there is a unique inner product on $H \otimes_{alg} K$ such that $(x_1 \otimes y_1, x_2 \otimes y_2) = (x_1, x_2) \cdot (y_1, y_2)$, where $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. The tensor product, $H \otimes K$, of Hilbert spaces H and K is defined as the Hilbert space resulting by completion of $H \otimes_{alg} K$ with respect to its inner product. Given two operators $a \in \mathcal{B}(H)$ and $b \in \mathcal{B}(K)$ there is a unique bounded operator on $H \otimes K$, denoted by $a \otimes b$, whose action on $H \otimes K$

is determined by $(a \otimes b)(\xi \otimes \nu) = a\xi \otimes b\nu$ for all $\xi \in H, \nu \in K$. It holds that $||a \otimes b|| = ||a|| \cdot ||b||$. Suppose now that $\mathcal{A}$ and B are two C^*-algebras and choose faithful representations π_1 and π_2 of $\mathcal{A}$ and B on the Hilbert spaces H and K, respectively. The C^*-subalgebra of $\mathcal{B}(H \otimes K)$ generated by the set $\{a \otimes b \,|\, a \in \pi_1(\mathcal{A}),\, b \in \pi_2(B)\}$ does not depend (as an abstract algebra) on the choice of the faithful representations π_1 and π_2 and it is called the *spatial tensor product* (or the *minimal tensor product*) of $\mathcal{A}$ and B.

Let φ_1 and φ_2 be states on the algebras $\mathcal{A}$ and B, respectively. There is a state φ, called the *product state* of φ_1 and φ_2 (in symbols $\varphi_1 \otimes \varphi_2$) that is uniquely determined by the condition $(\varphi_1 \otimes \varphi_2)(a \otimes b) = \varphi_1(a)\varphi_2(b)$ for all $a \in \mathcal{A}$ and $b \in B$. The product state $\varphi_1 \otimes \varphi_2$ is pure if, and only if, φ_1 and φ_2 are pure states.

Given two $*$-representations $\pi_1 : \mathcal{A} \mapsto \mathcal{B}(H)$ and $\pi_2 : B \mapsto \mathcal{B}(K)$, there is a unique *product representation* $\pi_1 \otimes \pi_2 : \mathcal{A} \otimes B \mapsto \mathcal{B}(H) \otimes \mathcal{B}(K)$ such that $(\pi_1 \otimes \pi_2)(a \ \otimes b) = \pi_1(a) \otimes \pi_2(b)$ for all $a \in \mathcal{A}$ and $b \in B$. The product $\pi_1 \otimes \pi_2$ is irreducible if, and only if, π_1 and π_2 are irreducible.

Let $P(\mathcal{A})$ denote the set of all projections in a C^*-algebra $\mathcal{A}$. Projections in e and f in a C^*-algebra $\mathcal{A}$ are called *orthogonal* if $ef = 0$. If e and f are orthogonal projections then the C^*-subalgebra generated by hereditary subalgebras eAe and fAf is $*$-isomorphic to their direct sum. The set $P(\mathcal{A})$ with the order inherited from $\mathcal{A}$ becomes partially ordered set. It holds that $e \leq f$ in $P(\mathcal{A})$ if, and only if, $ef = fe = e$. If $\mathcal{A}$ is unital, then the structure $P(\mathcal{A})$ is an orthomodular poset with the complement $p^\perp = \mathbb{I} - p$. Two projections $p, q \in P(\mathcal{A})$ are called (*Murray-von Neumann*) *equivalent* (in symbols $p \sim q$) if $p = v^*v$ and $q = vv^*$ for some $v \in \mathcal{A}$. The relation $\sim$ is an equivalence on the set $P(\mathcal{A})$. The projection p is said to be *subequivalent to a projection* q (in symbols $p \lesssim q$) if $p \sim e$, where $e \leq q$. All distinct projections in an abelian algebra are not equivalent, the projections in the algebra $\mathcal{B}(\mathcal{H})$ are equivalent if, and only if, their ranges have the same (orthonormal) dimension. If $||p - q|| < 1$ for $p, q \in P(\mathcal{A})$, then $p \sim q$. The projections p and q in a unital C^*-algebra $\mathcal{A}$ are said to be *unitarily equivalent* (in symbols $p \sim_u q$) if there is a unitary map $u \in \mathcal{A}$ such that $p = u^*qu$. It holds that $p \sim_u q$ if, and only if, $p \sim q$ and $\mathbb{I} - p \sim \mathbb{I} - q$. Orthogonal equivalent projections are always unitarily equivalent.

Suppose now that $e_1, \ldots, e_n$ are orthogonal equivalent projections in a unital C^*-algebra $\mathcal{A}$ with $e_1 + \cdots + e_n = \mathbb{I}$. Let v_{ij} be a partial isometry with the initial projection e_j and the final projection e_i. The system $(v)ij)$ is called the system of $n \times n$ *matrix units*. It holds that $v_{ij}v_{kl} = 0$ if $j \neq k$ and $v_{ij}v_{jk} = v_{ik}$.

A projection p in a C^*-algebra $\mathcal{A}$ is said to be *infinite* if there is a projection $q \in \mathcal{A}$ such that $p \sim q < p$. If p is not finite, then p is said to be *infinite*. A unital C^*-algebra is said to be *finite* (resp. *infitinie*) if its unit is a finite (resp. infinite) projection. A projection in $\mathcal{B}(\mathcal{H})$ is finite if, and only if, it is finite-dimensional. Hence, $\mathcal{B}(\mathcal{H})$ is a finite algebra if, and only if, $\dim\mathcal{H} < \infty$.

A state τ on C^*-algebra is called *tracial* (or a *trace*) if $\tau(x^*x) = \tau(xx^*)$ for all $x \in \mathcal{A}$. This is equivalent to the fact that $\tau(ab) = \tau(ba)$ for all $a, b \in \mathcal{A}$. The trace is constant on equivalent projections. There is only one tracial state on the matrix algebra $M_n(\mathbb{C})$ which is the standard normalized matrix trace.

A unital C^*-algebra is said to have *real rank zero* if every self-adjoint element can be approximated by self-adjoint elements with finite spectrum. Consequently, the real rank zero algebra is a closed linear span of its projections. A C^*-algebra $\mathcal{A}$ is called *monotone complete* if each bounded increasing net in $\mathcal{A}_{sa}$ has a least upper bound in $\mathcal{A}_{sa}$.

2.2 von Neumann algebras

Let us first mention some locally convex topologies on the algebra $\mathcal{B}(\mathcal{H})$. The *weak operator topology* on $\mathcal{B}(\mathcal{H})$ is given by the system of seminorms $a \in \mathcal{B}(\mathcal{H}) \mapsto |(ax, y)|$, where $x, y \in \mathcal{H}$. The convergence in this topology is the pointwise weak convergence. The *strong operator topology* is given by the system of seminorms $a \in \mathcal{B}(\mathcal{H}) \mapsto \|ax\|$, where $x \in \mathcal{H}$. The strong operator topology is the topology of pointwise norm convergence and it is finer then the weak operator topology.

Let $x \in \mathcal{B}(\mathcal{H})$ and $||x||_1 = \sum_{e\in E}(|x|e, e)$, where E is an orthonormal basis of $\mathcal{H}$ and $|x| = \sqrt{x^*x}$. The operator $x \in \mathcal{B}(\mathcal{H})$ is said to be the trace class operator if $||x||_1 < \infty$. The set of all trace class operators on $\mathcal{H}$ shall be denoted by $L^1(\mathcal{H})$.

The *σ-weak topology* (or *ultraweak topology*) is given by the system of seminorms induced by the trace class operators: $a \in \mathcal{B}(\mathcal{H}) \mapsto |tr(ta)|$, where $t \in L^1(\mathcal{H})$. The σ-weak topology is finer than the weak operator topology and both topologies coincide on the closed unit ball of $\mathcal{B}(\mathcal{H})$.

The *von Neumann algebra* (also W^*-algebra) is defined as the C^*-algebra that can be faithfully represented as a strongly operator closed $*$-subalgebra of $\mathcal{B}(\mathcal{H})$. The following conditions on a $*$-subalgebra M of $\mathcal{B}(\mathcal{H})$ are equivalent: (i) M is weakly operator closed (ii) M is strongly operator closed (iii) M is ultraweakly closed. It can also be proved that the commutant of any $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ is a von Neumann algebra. The so-called *von Neumann Bicommutant Theorem* says: a $*$-subalgebra M of $\mathcal{B}(\mathcal{H})$ that contains the unit of $\mathcal{B}(\mathcal{H})$ is a von Neumann algebra if, and only if, $M'' = M$. In other words, if M is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$, then every element in the bicommutant M'' can be approximated in the strong operator topology by element in M. Another very important approximation theorem is the *Kaplansky Density Theorem*:

Theorem 3 *Let $\mathcal{A}$ be a C^*-subalgebra of operators that generate a von Neumann algebra M. Then the unit ball of A is strongly operator dense in the unit ball of M.*

Let M be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$. The its *center*, $Z(M)$, is an abelian von Neumann subalgebra of M and $Z(M) = M \cap M'$. A von Neumann algebra is called the *factor* if $Z(M)$ consists of scalar multiples of the unit of M only. For each projection $p \in P(M)$ we define the *central cover*, $c(p)$, of p as the smallest central projection majorizing p. If z is a central projection and $p \in P(M)$, then $c(zp) = zc(p)$.

Any weakly operator closed left ideal $\mathcal{I}$ in a von Neumann algebra M is of the form $\mathcal{I} = Mp$, where p is a uniquely determined projection. Any weakly operator closed ideal in M is of the form $\mathcal{I} = zM$, where z is a uniquely determined central projection in M. All weakly operator closed hereditary subalgebras of M are of the form $pMp = \{pxp \mid x \in M\}$, where p is a uniquely determined projection. If we consider the algebra pMp as acting on the Hilbert space $p(\mathcal{H})$, then the following equalities hold: $(pMp)' = pM'p$ and $Z(pMp) = pZ(M)p$. This implies that any hereditary subalgebra of a factor is a factor, too.

Now consider M to be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$. A bounded functional φ on M is called *normal* if $\varphi(u_\alpha) \to \varphi(u)$ whenever $u_\alpha \nearrow u$ in M_{sa}. It turns out that the following conditions are equivalent: (i) φ is a normal functional (ii) φ is weakly operator continuous on the unit ball of M (iii) φ is strongly operator continuous on the unit ball of M (iv) φ is a completely additive measure on the projection lattice $P(M)$, i.e. $\varphi\big(\sum_\alpha p_\alpha\big) = \sum_\alpha \varphi(p_\alpha)$ for any system of pairwise orthogonal projections (p_α) in M (v) φ is continuous in the ultraweak topology.

Any normal functional on M is given by a trace class operator $t \in L^1(\mathcal{H})$ such that $\varphi(x) = tr(tx)$ for each $x \in M$.

Let $M_1, \ldots, M_n$ be von Neumann algebras acting on the Hilbert spaces $\mathcal{H}_1, \ldots, \mathcal{H}_n$, correspondingly. The *von Neumann tensor product* of algebras $M_1, \ldots, M_n$, denoted by $M_1 \overline{\otimes} \cdots \overline{\otimes} M_n$ is the von Neumann subalgebra of $\mathcal{B}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n)$ generated by the spatial C^*-tensor product $M_1 \otimes \cdots \otimes M_n$.

Suppose that $\varphi_1, \ldots, \varphi_n$ are normal states on von Neumann algebras $M_1, \ldots, M_n$, correspondingly. Then there is a unique normal state φ on $M_1 \overline{\otimes} \cdots \overline{\otimes} M_n$ such that $\varphi(x_1 \otimes \cdots \otimes x_n) = \varphi_1(x_1) \cdots \varphi_n(x_n)$ for all $x_1 \in M_1, \ldots, x_n \in M_n$. Such state φ is called the *normal product state* and it is denoted by the symbol $\varphi_1 \overline{\otimes} \cdots \overline{\otimes} \varphi_n$.

If M and N are von Neumann algebras acting on Hilbert spaces $\mathcal{H}$ and K, respectively, then $(M \overline{\otimes} N)' = M' \overline{\otimes} N'$. In particular, $Z(M \overline{\otimes} N) = Z(M) \overline{\otimes} Z(N)$. Hence, product of factors is a factor, too.

The projection lattice $P(M)$ is a complete orthomodular lattice. By $e \vee f$ and $e \wedge f$ we shall denote the supremum and the infimum of the projections e and f in $P(M)$, respectively.

Let $\mathcal{A}$ be a C^*-algebra. A vector $\psi \in \mathcal{H}$ is called cyclic for the representation π of $\mathcal{A}$ on Hilbert space $\mathcal{H}$ if the set $\pi(\mathcal{A})\psi = \{\pi(a)\psi \,|\, a \in \mathcal{A}\}$ is dense in $\mathcal{H}$.

Besides the order inherited from the C^*-algebra, the structure of projections is endowed with an equivalence relation, $\sim$. The subequivalence relation introduces another order on projections reflecting their dimensions. It holds that $e \sim f$ if, and only if, $e \lesssim f$ and $f \lesssim e$. If $e, f \in M$, then $e \vee f - f \sim e - e \wedge f$.[2].

Now let us classify various types of von Neumann algebras. A von Neumann algebra is said to be *Type I* algebra if there is an abelian projection $e \in M$ such that $c(e) = \mathbb{I}$. This is equivalent to saying that each nonzero central projection in M majorizes a nonzero abelian projection. Let n be a cardinal number. If the unit in a von Neumann algebra M can be written as a sum of n equivalent abelian projections, then M is said to be of *Type I_n*. Type I_1 algebras are just abelian von Neumann algebras. Each algebra of Type I_n is called *homogenous Type I algebra*. For every von Neumann algebra M of Type I and each cardinal less then card$M$ there is a uunique central projection z_α such that $z_\alpha M$ is either zero or Type I_α and such that $\sum_\alpha z_\alpha = \mathbb{I}$. Type I_α algebras are exactly the $*$-isomorphic copies of the tensor products $\mathcal{A} \overline{\otimes} \mathcal{B}(\mathcal{H}_\alpha)$, where $\mathcal{A}$ is an abelian von Neumann algebra and $\mathcal{H}_\alpha$ is a Hilbert space of dimension α. Type I factors are the algebras of all bounded operators on Hilbert spaces. Type I algebra is finite if, and only if, it is a direct sum of (countably many) Type I_n, $n < \infty$ algebras. The finite Type I homogenous algebras can be identified with the algebras $M_n(C(X))$, where X is a hyperstonean space. Another way of looking at this algebra is to represent it as algebra $C\big(X, M_n(\mathbb{C})\big)$ of all continuous functions on a hyperstonean space X. The algebras of Type I contain many abelian hereditary subalgebras, thus are close to abelian von Neumann algebras. Now we will summarize other types of von Neumann algebras - those without nonzero abelian part. A von Neumann algebra is said to be of *Type II* if it has no nonzero abelian projection but has a finite projection e such that $c(e) = \mathbb{I}$. Equivalently, the algebra M is said to be of Type II if it has no

[2]This relation is referred to as the *Kaplansky formula* or the *parallelogram law*

nonzero abelian projection and if every nonzero central projection majorizes a nonzero finite projection. The finite algebras of Type II are called *Type* II_1, the infinite ones are called *Type* II_∞. A von Neumamm algebra is called *semifinite* if it is a Type I or Type II or direct sum of algebras of these types. If M is properly infinite and semifinite von Neumann algebra, then there exists an orthogonal family (z_α) of central projections in M indexed by infinite cardinals less then card$M$ with $\sum_\alpha z_\alpha = \mathbb{I}$, and a family (N_α) of finite von Neumann algebras such that $z_\alpha M$ is $*$-isomorphic to $N_\alpha \overline{\otimes} \mathcal{B}(\mathcal{H}_\alpha)$, where $\dim H_\alpha = \alpha$ and z_α may be zero. The family (z_α) is uniquely determined.

Let M be a von Neumann algebra with the center $Z(M)$. The *center-valued trace* $T : M \mapsto Z(M)$ is a linear mapping such that (i) $T(xy) = T(yx)$ for all $x, y \in M$ (ii) T is identity on $Z(M)$ (iii) $T(x) > 0$ whenever $x \in M_+$ is positive. If such a mapping exists, then M has to be finite. Conversely, any finite von Neumann algebra admits exactly one center-valued trace, T. The mapping T has these additional properties: (iv) $T(zx) = zT(x)$ for all $z \in Z(M)$ and $x \in M$ (v) $T(e) \leq T(f)$ for projections e, f if, and only if, $e \precsim f$ (vi) $||T|| \leq 1$ (vii) T is ultraweakly continuous.

Let $T : M \mapsto Z(M)$ be a center-valued trace on M. Then the following holds: (i) if M is of Type I_n, then $T\big(P(M)\big)$ consists of all elements of the form $\frac{1}{n}z_1 + \frac{2}{n}z_2 + \cdots + \frac{n-1}{n}z_{n-1} + z_n$, where $z_1, z_2, \ldots, z_n$ are pairwise orthogonal central projections. (ii) If M is of Type II_1, then $T\big(P(M)\big)$ consists of all positive elements in the unit ball of $Z(M)$.

Type I_n factors are just matrix algebras $M_n(\mathbb{C})$. In this case the center-valued trace is nothing but the normalized matrix trace attaining discrete values $\frac{k}{n}$, $0 \leq k \leq n$, on projections. The faithful tracial state on Type II_1 factor attains all values in the interval $[0, 1]$ when restricted to projections. We can view Type II_1 factors as "continuous matrix algebras".

A von Neumann algebra is said to be of *Type* III (or *purely infinite*) if it contains no nonzero finite projection.

Every von Neumann algebra is uniquely decomposable into the direct sum of algebras of Type I, II_1, II_∞ and III. If von Neumann algebra has zero Type I finite part, then for every projection e in M and any integer n there are orthogonal equivalent projections $e_1, \ldots, e_n$ such that $e = e_1 + e_2 + \cdots + e_n$. In particular, for any projection e in arbitrary von Neumann algebra there are projections e_1, e_2 and e_3 such that $e = e_1 + e_2 + e_3$, $e_1 \sim e_2$ and e_3 is abelian.

2.3 Jordan Algebras

A *Jordan algebra* is a real algebra $(\mathcal{A}, \circ)$ such that the product $\circ$ has the following properties: (i) $a \circ b = b \circ a$ (ii) $a \circ (b \circ a^2) = (a \circ b) \circ a^2$, for all $a, b \in \mathcal{A}$. A *JB algebra* is a Jordan algebra $(\mathcal{A}, \circ, ||\cdot||)$ where the complete norm $||\cdot||$ satisfies the following conditions: (i) $||a \circ b|| \leq ||a|| \cdot ||b||$ (ii) $||a^2|| = ||a||^2$ (iii) $||a^2|| \leq ||a^2 + b^2||$, for all $a, b \in \mathcal{A}$. The important example of JB algebras are self-adjoint parts of C^*-algebras endowed with the anticommutant product $a \circ b = \frac{1}{2}(ab + ba)$.
Let R be any algebra. Then $M_n(R)$, the space of $n \times n$ matrices with coefficients in R, is also an algebra with the usual matrix product. The Hermitian, or self-adjoint, part of $M_n(R)$ will be denoted by $\mathbb{H}(R)$ and called a *Jordan matrix algebra*. Jordan matrix algebras of our main concern to us will be $\mathbb{H}_n(\mathbb{R})$, $\mathbb{H}_n(\mathbb{C})$, $\mathbb{H}_n(\mathbb{H})$ and $\mathbb{H}_n(\mathbb{O})$, where $\mathbb{H}$ and $\mathbb{O}$ is the algebra of quaternions or octonions, respectively.

Let $\mathcal{A}$ be a JB algebra. $\mathcal{A}$ is said to be *monotone complete* if each bounded increasing net (a_α) in $\mathcal{A}$ has a least upper bound $a \in \mathcal{A}$. A bounded linear functional ρ on $\mathcal{A}$ is called *normal* if $\rho(a_\alpha) \to \rho(a)$ for each net (a_α) as above. $\mathcal{A}$ is said to be a *JBW algebra* if $\mathcal{A}$ is monotone complete a has a separating set of positive normal bounded linear functionals. We call a set of functionals *separating* if for any nonzero $a \in \mathcal{A}$ there is a functional ρ in the set satisfying $\rho(a) \neq 0$.

Very important result in the theory of JBW algebras is that each JBW algebra is unital. JBW algebra $W(a)$ generated by an element a and $\mathbb{I}$ in a JBW algebra M is associative.

Analogously, a projection in a JBW algebra is a self-adjoint idempotent. Two projections p and q are said to be orthogonal if $p \circ q = 0$.

A *multiplication operator* is defined by $T_a b = a \circ b$ for all $a, b \in \mathcal{A}$. Two elements a and b in a Jordan algebra $\mathcal{A}$ are said to *operator commute* if the operators T_a and T_b commute, i.e. if $(a \circ c) \circ b = a \circ (c \circ b)$ for all $c \in \mathcal{A}$. The center of a Jordan algebra $\mathcal{A}$ is the set of all elements that operator-commute with all elements in $\mathcal{A}$. A symmetry in $\mathcal{A}$ is an operator s such that $s^2 = \mathbb{I}$.

Let $\mathcal{A}$ be a JBW algebra with the center Z. If p is a projection in $\mathcal{A}$ then its *central support* $c(p)$ is the smallest projection in Z majorizing p.

JBW algebras can be sorted out among different types just like in the case of von Neumann algebras.

Define operator $U_{a,c}$ by $U_{a,c}(b) = \{abc\}$ and $U_s(a) = \{sas\}$ for some a, b, c and s in a Jordan algebra.

Let M be a JBW algebra. If $s \in M$ is a symmetry, then U_s is an automorphism of M. These automorphisms generate a group $\mathrm{Int}M$, called the group of *inner automorphisms* of M.

Two projections p and q in M are called *equivalent* if there is a $\alpha \in \mathrm{Int}M$ such that $q = \alpha(p)$. We then write $p \sim q$. If α can be written as $\alpha = U_{s_1}U_{s_2}\dots U_{s_n}$ we write $p \sim_n q$. If $n = 1$ we say p and q are *exchanged by a symmetry*. In contrast to the equivalence of projections in a von Neumann algebra, it holds that if $p \sim q$ then $p^\perp \sim q^\perp$.

A projection $p \in M$ is called *Abelian* if M_p is associative; p is modular if the projection lattice $[0, p]$ of M_p is modular. If $\mathbb{I}$ is modular, M itself is called modular. The set of Abelian and modular projections are $\mathrm{Int}M$ invariant, so we can define central projections e_I and e_{iii} in M by: $e_I = \bigvee\{p \in M : p \,\text{is Abelian}\}$ and $e_{III}^\perp = \bigvee\{p \in P : p \text{ ismodular}\}$.

Now let $e_{II} = \mathbb{I} - e_I - e_{III}$. M is said to be of *Type I* (resp. II, III) if $e_I = \mathbb{I}$ (resp. $e_{II} = \mathbb{I}$, $e_{III} = \mathbb{I}$).

Theorem 4 *Let M be a JBW algebra. Then M can be split uniquely into a direct sum of parts of Type I, II and III, the different parts being characterized as follows:*

(i) M is of Type I if and only if there is an Abelian projection $p \in M$ with $c(p) = \mathbb{I}$.

(ii) M is of Type II if and only if there is a modular projection $p \in M$ with $c(p) = \mathbb{I}$, and M contains no nonzero Abelian projection.

(iii) M is of Type III if and only if it contains no nonzero modular projection.

Proof can be found in [14].
Just like in the case of von Neumann algebras we can define a finer decomposition. Any JBW algebra has a largest central modular projection. If this is 0, M is called *purely nonmodular*; if it is $\mathbb{I}$, M is modular. We say M is of type II_1 if it is modular and is of type II; we say that M is of type II_∞ if it is purely nonmodular and of type II. If M is of type III, it is purely nonmodular.
indent We call two orthogonal projections p, q in Jordan algebra M *strongly connected* if there exists $v \in \{pMq\}$ such that $v^2 = p + q$.

Lemma 5 *Let M be a JBW algebra with projection lattice P. Suppose M has no direct summand of Type I. Then there is $p \in P$ with $p \sim_1 p^\perp$.*

Proposition 6 *Let M be a JBW algebra with projection lattice P. Suppose M has no direct summand of Type I. Then there are $p_i \in P$, $i = 1, \dots, 4$, such that $p_1 + \dots + p_4 = \mathbb{I}$, and $p_i \sim_1 p_j$ for all i, j.*

Let M be a JBW algebra and n a cardinal number. We say M is of Type I_n if there is a family $(p_\alpha)_{\alpha\in J}$ of Abelian projections such that $c(p_\alpha) = \mathbb{I}$, $\sum_{\alpha\in J} p_\alpha = \mathbb{I}$ and card$J = n$. M is of Type I_∞ if M is a direct sum of JBW algebras of Type I_n with n infinite.

Theorem 7 *Each JBW algebra of Type I has a unique decomposition*

$$M = M_1 \oplus M_2 \oplus \cdots \oplus M_\infty,$$

where each M_n is either 0 *or is a JBW algebra of Type I_n.*

Proof can be found in [14].

Theorem 8 *Let M be a JBW algebra with projection lattice $\mathcal{P}$. Suppose M has no direct summand of Type I. Then there is $p \in \mathcal{P}$ with $p \sim_1 p^\perp$.*

Proof can be found in [14].

Let M be a JBW factor of Type I_n, $3 \le n < \infty$. Then M is isomorphic to one of $\mathbb{H}_n(\mathbb{R})$, $\mathbb{H}_n(\mathbb{C})$, $\mathbb{H}_n(\mathbb{H})$ or $\mathbb{H}_n(\mathbb{O})$ in the case $n = 3$.

Proposition 9 *Let M be a JBW algebra with projection lattice P, and let J be an index set. Let $p, q, p_\alpha, q_\alpha \in P$, $\alpha \in J$, satisfy $p \perp q$, $p = \sum_{\alpha\in J} p_\alpha$, $q = \sum_{\alpha\in J} q_\alpha$ and $p_\alpha \sim_1 q_\alpha$ for all $\alpha \in J$. Then $p \sim_1 q$.*

Very significant role is played by so-called spin factors. Let B be a real unital Jordan algebra. A *spin system* in B is a collection $\mathcal{P}$ of at least two symmetries different from $\pm\mathbb{I}$ such that $s \circ t = 0$ whenever $s \neq t$ in $\mathcal{P}$. A unital JB algebra generated as a JB algebra by a spin system is called *spin factor*. For each cardinal number $n \ge 2$ there is, up to isomorphism, a unique spin system generated by a spin system of cardinality n. JBW algebra is a JBW factor of Type I_2 if and only if it is a spin factor.

Symmetric matrix algebra $\mathbb{H}_2(\mathbb{R})$ is a spin factor. Basis for $\mathbb{H}_2(\mathbb{R})$ is

$$\mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where $\sigma_i^2 = \mathbb{I}$, $i = 1, 2$ and $\sigma_1 \circ \sigma_2 = 0$. These matrices are so-called Pauli spin matrices.

2.4 Basic axioms of C^*-algebraic quantum mechanics

After having reviewed the theory of operator algebras, we will now summarize basic axioms of C^*-algebraic quantum mechanics:

(i) The set of all observables of a quantum system S is the self-adjoint part of a C^*-algebra $\mathcal{A}$.

(ii) The set of all states of a quantum system is the state space, denoted by $S(\mathcal{A})$, of the C^*-algebra $\mathcal{A}$.

(iii) The value $\rho(a)$, where $\rho \in S(\mathcal{A})$ and $a \in \mathcal{A}_{sa}$ is the expectation value of an observable a on the condition that a system S is prepared in the state ρ.

(iv) Evolution of a system S is given by specified class of morphisms of the C^*-algebra $\mathcal{A}$ (unitary maps, automorphisms).

(v) Given independent quantum systems S_1 and S_2 represented by C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively, the smallest composite system containing S_1 and S_2 is given by the minimal tensor product $\mathcal{A} \otimes_{min} \mathcal{B}$.

Chapter 3

Quantum history approach

In physics we often want to describe "degree of interference" between observables a and b or correlation between random variables a and b. This can be achieved by virtue of a bilinear form $F(a, b)$. Coming to higher dimensions let us suppose that the quantum system is represented by the algebra $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space, and by a normal state $\rho \in \mathcal{B}(\mathcal{H})^*$ with the associated tracial operator t_ρ. Let $p_{t_1}, p_{t_2}, \ldots, p_{t_n}$ be a finite sequence of projections labeled by a discrete set of time parameters $\{t_1, t_2, \ldots, t_n\}$ that represents a series of possible measurement outcomes. Such a sequence is called the *homogenous history of order* n. In the Heisenberg picture it can be seen as a development of the initial two-valued observable p_{t_1} and the projections p_{t_i} depend on the Hamiltonian of the system

$$p_{t_i} = e^{i(t_i - t_1)H} p_1 e^{-i(t_i - t_1)H},$$

where H is the Hamiltonian operator. The probability $d_\rho(p_{t_1}, p_{t_2}, \ldots, p_{t_n})$ to obtain the history $p_{t_1}, p_{t_2}, \ldots, p_{t_n}$ is given by *Wigner formula*

$$\overline{d}_\rho(p_{t_1}, p_{t_2}, \ldots, p_{t_n}) = \rho(p_{t_1} p_{t_2} \cdots p_{t_n} p_{t_{n-1}} p_{t_{n-2}} \cdots p_{t_1}),$$

or, in a physically more popular form,

$$\overline{d}_\rho(p_{t_1}, p_{t_2}, \ldots, p_{t_n}) = tr(p_{t_n} p_{t_{n-1}} \cdots p_{t_1} t_\rho p_{t_1} p_{t_2} \cdots p_{t_n}),$$

which can be naturally associated with a multimeasure, d_ρ, with $2n$ arguments

$$d_\rho(h_{t_1}, h_{t_2}, \ldots, h_{t_n}, k_{t_1}, k_{t_2}, \ldots, k_{t_n}) = tr(h_{t_1} h_{t_2} \cdots h_{t_n} t_\rho k_{t_1} k_{t_2} \cdots k_{t_n}),$$

which represents the "influence" between histories $h = \{h_1, h_2, \ldots, h_n\}$ and $k = \{k_1, k_2, \ldots, k_n\}$. Taking $h = k$ we get the probability of obtaining the history h. The map d_ρ called the *standard decoherence functional* is a motivation in multimeasures studies. While generalized Gleason Theorem establishes the one-to-one correspondence between bounded measures on projection lattices of von Neuman algebras and linear functional we may question ourselves whether there exists a similar correspondence between functions on products of projection lattices that are separately finitely additive and multilinear maps on the products of the corresponding algebras.

In the following section we shall establish necessary mathematical apparatus before proceeding further to decoherence functionals.

Definition 10 *Let $A_1, A_2, \dots, A_n$ be C^*-algebras with the projection structures $P(A_1), \dots, P(A_n)$, respectively. Let X be a Banach space. The map $m : P(A_1) \times P(A_2) \times \dots \times P(A_n) \mapsto X$ is called an X-*valued multimeasure *(in short* multimeasure*) if m is separately finitely additive, meaning that, for each j,*

$$m(p_1, \dots, p_{j-1}, q_1 + q_2, p_{j+1}, \dots, p_n) = \\ = m(p_1, \dots, p_{j-1}, q_1, p_{j+1}, \dots, p_n) + m(p_1, \dots, p_{j-1}, q_2, p_{j+1}, \dots, p_n), \quad (3.1)$$

where $p_i \in P(A_i)$, q_1 and q_2 are orthogonal projections in $P(A_j)$. If $X = \mathbb{C}$, we call m the complex multimeasure. Completely additive *and* σ-additive *multimeasures are defined in the standard way as measures separately completely additive and σ-additive, respectively. In case when $n = 2$ we call the multimeasure the* bimeasure. *Moreover, if $A = A_1 = A_2$ we say that m is a bimeasure on A. A bimeasure m on A is said to be* hermitian *if*

$$m(p, q) = \overline{m(q, p)}$$

for all projections $p, q \in P(A)$.

Definition 11 *Let $X, X_1, X_2, \dots, X_n$ be Banach spaces. Let $F : X_1 \times X_2 \times \dots \times X_n \mapsto X$ be an n-linear map. F is said to be* bounded *if there is a constant C such that, for each $x_i \in X_i$,*

$$||F(x_1, x_2, \dots, x_n)|| \leq C||x_1|| \cdot ||x_2|| \cdots ||x_n||.$$

A smallest constant C for which this inequality holds is said to be the norm *of F (in symbols $||F||$).*

The next theorem gives an answer to our motivating question whether there exists generalization of Gleason Theorem [11] to multimeasures.

Theorem 12 *Let $M_1, M_2, \dots, M_n$ be von Neumann algebras, each having no direct summand of Type I_2. Suppose that $m : P(M_1) \times P(M_2) \times \dots \times P(M_n) \mapsto X$ is a bounded X-valued multimeasure. There is a unique bounded n-linear map $F : M_1 \times M_2 \times \dots \times M_n \mapsto X$ which extends m.*

Proof can be found in [11].

Theorem 13 *Let M_1 and M_2 be von Neumann algebras without any Type I_2 direct summand and any finite-dimensional direct summand. Suppose that $m : P(M_1) \times P(M_2) \mapsto \mathbb{C}$ is a completely additive bimeasure. Then m is bounded. Furthermore, there exists a bounded linear form, $F : M_1 \times M_2 \mapsto \mathbb{C}$, which extends m and, moreover, both $F(x, \cdot)$ and $F(\cdot, y)$ are normal functionals for all $x \in M_1$ and $y \in M_2$.*

Proof can be found in [11].

The general quantum histories theory can be characterized by the space of *histories* and space of *decoherence functionals*. The histories generalize the time ordered strings of projections, while decoherence functionals describe the probabilistic correlations between various histories.

Definition 14 *Let $M_1, M_2, \ldots, M_n$ be von Neumann algebras. A* decoherence functional *(of order n) is a complex multimeasure*

$$d : P(M_1) \times P(M_2) \times \cdots \times P(M_n) \times P(M_1) \times P(M_2) \times \cdots \times P(M_n) \mapsto \mathbb{C}$$

such that

(i) $d(p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n) = \overline{d(q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n, q_1)}$ *for all p_i and q_i in $P(M_i)$ (hermiteanetness).*

(ii) $d(p_1, p_2, \ldots, p_n, p_1, p_2, \ldots, p_n) \geq 0$ *for all $p_i \in P(M_i)$ (positivity).*

(iii) $d(\mathbb{I}, \mathbb{I}, \ldots, \mathbb{I}, \mathbb{I}, \ldots, \mathbb{I}) = 1$ *(normalization).*

The decoherence functional is called completely additive *if it is a completely additive multimeasure. If all algebras $M_1, M_2, \ldots, M_n$ equal to the algebra M, we say that d is the* decoherence functional *(of order n) on M.*

Corollary 15 *Let $d : P(M_1) \times P(M_2) \times \cdots \times P(M_n) \times P(M_1) \times P(M_2) \times \cdots \times P(M_n) \mapsto \mathbb{C}$ be a bounded decoherence functional such that all von Neumann algebras $M_1, M_2, \ldots, M_n$ have no Type I_2 direct summand. Then there is a unique bounded multilinear form*

$$F : M_1 \times M_2 \times \cdots \times M_n \times M_1 \times M_2 \times \cdots \times M_n \mapsto \mathbb{C}$$

which extends d.

The set of all homogenous histories of order n has a priori no structure of propositional logic. However, we can identify each homogenous history $h = (h_1, h_2, \ldots, h_n)$ with the simple tensor $h_1 \otimes h_2 \otimes \cdots \otimes h_n$ in the n-fold tensor product $B(\mathcal{H}) \overline{\otimes} \cdots \overline{\otimes} B(\mathcal{H})$ which endows homogenous histories with the structure of propositions. Therefore we can consider *general history* as the (general) projection in the lattice $P\big(B(\mathcal{H})\overline{\otimes}\cdots\overline{\otimes}B(\mathcal{H})\big)$. Previously we dealt with extending functionals from projection lattice of an algebra to the whole algebra. In this connection a natural question arises whether also the decoherence functional could be expressed in terms of the tensor product and viewed as extended from homogenous histories to all general histories. The following theorem states this is possible in finite dimensions.

Theorem 16 ***(Isham-Linden-Schreckenberg)*** *Let $\mathcal{H}$ be a finite-dimensional Hilbert space of dimension at least three. Let d be a bounded decoherence functional of order n on $\mathcal{B}(\mathcal{H})$. Then there is a unique operator X_d on the tensor product $K \otimes K$, where $K = \mathcal{H} \otimes \mathcal{H} \otimes \cdots \otimes \mathcal{H}$ is the n-fold tensor product such that*

$$d(p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n) = tr\big((p \otimes q)X_d\big), \tag{3.2}$$

where $p = p_1 \otimes p_2 \otimes \cdots \otimes p_n$ and $q = q_1 \otimes q_2 \otimes \cdots \otimes q_n$.

Proof. By our assumption and the previous Theorem such bounded decoherence functional uniquely extends to a bounded multilinear form $F : \bigotimes_{j=1}^{2n} \mathcal{H} \mapsto \mathbb{C}$. There exists unique linear form f on $K \otimes_{alg} K$ such that

$$d(p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n) = f(p_1 \otimes p_2 \otimes \cdots \otimes p_n \otimes q_1 \otimes q_2 \otimes \cdots \otimes q_n)$$

for all $p_i, q_i \in P(\mathcal{H})$. The fact that any linear form on finite-dimensional Hilbert space is normal completes the proof. □

The formula (3.2) is reffered to as the *Isham-Linden-Schreckenberg representation* and can be viewed as an analog of Gleason Theorem (for more details see [11]) to quantum histories in finite dimensions. Any bounded decoherence functional in uniquely represented by a linear form on the corresponding algebraic tensor product. However, this form does not have to be continuous, even if the decoherence functional is completely additive and bounded. Example exhibiting this is in [11].

If μ is a completely additive probability measure on the projection lattice of an infinite-dimensional Hilbert space $\mathcal{H}$, then μ extends to a normal functional which is described by the trace class operator on $\mathcal{H}$. This representation corresponds to a decomposition of μ into convex mixture of pure states. The relation between completely additive measures and trace class operators breaks down for completely additive bimeasures. It indicates the different geometric structure of decoherence functionals. For details and examples, see [11].

It turns out that the boudedness of the corresponding linear form on compact operators is necessary and sufficient for the existence of the Isham-Linden-Schreckenberg representation.

Now we shall use the following conventions. Let $K(\mathcal{H})$ be the ideal of compact operators on the Hilbert space $\mathcal{H}$. The algebraic tensor product $K(\mathcal{H}) \otimes_{alg} K(\mathcal{H})$ will be viewed as subalgebra of $\mathcal{B}(\mathcal{H} \otimes \mathcal{H})$. By [11], pg. 220, it can be proved that $K(\mathcal{H}) \otimes K(\mathcal{H}) = K(\mathcal{H} \otimes \mathcal{H})$.

Definition 17 *We say that the decoherence functional d of order one on $\mathcal{B}(\mathcal{H})$ is* tensor bounded *if the corresponding linear form f_d on $\mathcal{B}(\mathcal{H}) \bigotimes_{alg} \mathcal{B}(\mathcal{H})$ is bounded when restricted to $K(\mathcal{H}) \bigotimes_{alg} K(\mathcal{H})$.*

Theorem 18 *Let $\mathcal{H}$ be a Hilbert space of dimension dim $\mathcal{H} \geq 3$. Let d be a bounded decoherence functional of order one on $\mathcal{B}(\mathcal{H})$. Then d is tensor bounded if, and only if, there exists a trace class operator X_d on $\mathcal{H} \bigotimes \mathcal{H}$ such that*

$$d(p,q) = tr\big((p \otimes q)X_d\big) \tag{3.3}$$

for all finite-dimensional projections $p, q \in \mathcal{B}(\mathcal{H})$.

Corollary 19 *Let H be a Hilbert space with dim $H \geq 3$. Let d be a completely additive decoherence functional on $\mathcal{B}(\mathcal{H})$ of order one. There exists a trace class operator X_d on $\mathcal{H} \otimes \mathcal{H}$ such that*

$$d(p,q) = tr\big((p \otimes q)X_d\big)$$

for all projections $p, q \in \mathcal{B}(\mathcal{H})$ if, and only if, d is tensor bounded.

Proof of Theorem (18) and Corollary (19) can be found in [11].

Although completely additive decoherence functionals cannot in general be described by the trace class operators, it can be in many important cases computed from the formula (3.2). To formulate this fact we shall need the following concept:

Definition 20 *Let d be a decoherence functional on $\mathcal{B}(\mathcal{H})$ of order one with the corresponding linear form f_d on $\mathcal{B}(\mathcal{H}) \otimes_{alg} \mathcal{B}(\mathcal{H})$. Let us say that d is* tracially bounded *if*

$$sup\{|f_d(p_\xi)|\,|\,\xi \in \mathcal{H} \otimes_{alg} \mathcal{H},\ ||\xi|| = 1\} < \infty\,,$$

where p_ξ denotes the projection of $\mathcal{H} \otimes \mathcal{H}$ onto the one-dimensional space $sp\{\xi\}$.

Theorem 21 *Let the decoherence functional d be of order one on the algebra $\mathcal{B}(\mathcal{H})$, dim $\mathcal{H} \geq 3$. Let d be tracially bounded. Then there is a bounded operator T on $\mathcal{H} \otimes \mathcal{H}$ such that*

$$d(p,q) = tr\big((p \otimes q)T\big)\,,$$

whenever p and q are finite-dimensional projections on $\mathcal{H}$.

Furthermore, let d be completely additive. Then for all projections $p, q \in$

$\mathcal{B}(\mathcal{H})$ we have

$$d(p,q) = \sum_i \sum_j tr\big((p_i \otimes q_j)T\big)\,, \tag{3.4}$$

whenever $p = \sum_i p_i$ and $q = \sum_j q_j$, where p_i, q_j are one-dimensional projections in $\mathcal{H}$.

Proof can be found in [11].

3.1 Local operations and measurements

In this paragraph we shall show application of the multi-form Gleason Theorem in quantum information theory. Let $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$ be the tensor product of Hilbert spaces with the set of all unit vectors denoted by $\sum(\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n)$ of the form $x_1 \otimes x_2 \otimes \cdots \otimes x_n$, where $x_i \in \mathcal{H}_i$ are unit vectors.

A natural question in quantum measurement theory is whether the knowledge of structure of probabilities associated to local measurements allows one to determine uniquely the state of the whole system.

Let us represent probabilities on local measurements by the function

$$f : \sum(\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n) \mapsto \mathbb{R}^+$$

with the property

$$\sum_{i \in I} f(e_i) = w,$$

whenever $(e_i)_{i\in I}$ is an orthonormal basis of $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$ contained in $\sum(\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n)$. The function f is called *unentangled frame function* with the *weight* w. Mathematical part of the local measurements is whether unentangled frame function can be extended to the frame function on the whole of $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$. The following proposition says that we can always find a multimeasure representing an unentangled frame function.

Proposition 22 *Let $f : \sum(\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n) \mapsto \mathbb{R}^+$ be an unentangled frame function. Then there is a completely additive bounded multimeasure m on $P(\mathcal{H}_1) \times P(\mathcal{H}_2) \times \cdots \times P(\mathcal{H}_n)$ such that*

$$f(x_1 \otimes \cdots \otimes x_n) = m(p_1, p_2, \ldots, p_n),$$

whenever $x_1 \otimes \cdots \otimes x_n \in \sum(\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n)$ and $p_i\,(i = 1, \ldots, n)$ is the projection with the range $sp\{x_i\}$.

Proof. Can be found in [11]. □

Theorem 23 *Let $\mathcal{H}_1, \ldots \mathcal{H}_n$ be Hilbert spaces of dimension at least three. Let $f : \sum(\mathcal{H}_1, \ldots, \mathcal{H}_n) \mapsto \mathbb{R}^+$ be an unentangled frame function such that the corresponding multimeasure is tensor bounded. Then there is a trace class operator X on $B(\mathcal{H}_1, \ldots, \mathcal{H}_n)$ such that*

$$f(x_1 \otimes \cdots \otimes x_n) = tr\big((p_1 \otimes \cdots \otimes p_n)X\big),$$

for all unit vectors $x_i \in \mathcal{H}_i$ and corresponding one-dimensional projections p_i, $i = 1, \ldots, n$.

Proof can be found in [11].

Corollary 24 ***(Wallach Theorem)*** *Let $\mathcal{H}_1, \ldots \mathcal{H}_n$ be finite-dimensional Hilbert spaces, each of dimension at least three. Let $f : \sum(\mathcal{H}_1, \ldots \mathcal{H}_n) \mapsto \mathbb{R}^+$ be an unentangled frame function. Then there exists a self-adjoint operator $T \in \mathcal{B}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n)$ such that whenever $x_1 \otimes \cdots \otimes x_n$ is in $\sum(\mathcal{H}_1, \ldots \mathcal{H}_n)$ and p_j is the projection of $\mathcal{H}_j$ onto the one-dimensional subspace generated by x_j, the following equality is in force:*

$$f(x_1 \otimes x_2 \otimes \cdots \otimes x_n) = tr\big((p_1 \otimes p_2 \otimes \cdots \otimes p_n)T\big).$$

Chapter 4

Hidden variables

4.1 Hidden Variables in von Neumann Algebras

Definition 25 *Let L be an orthomodular lattice. By a dispersion-free state on L we mean a finitely additive probability measure on L with values in the set $\{0, 1\}$.*

Theorem 26 (Hamhalter, (1993)) *The projection lattice $P(M)$ of a von Neumann algebra M which has neither a non-zero abelian nor a type I_2 direct summand admits no dispersion-free state.*

The following proposition reduces the investigation of dispersion-free states to simple matrix algebras instead of applying the Gleason Theorem for von Neumann algebras of infinite dimension.

Theorem 27 *Let M be a von Neumann algebra with no non-zero abelian direct summand and no type I_2 direct summand. The following statements hold:*

(i) Any subalgebra of M which is $*$-isomorphic to $M_2(\mathbb{C})$ is contained in a subalgebra $C \oplus D$ of M satisfying the following properties: C is either zero or it is $*$-isomorphic to $M_4(\mathbb{C})$; D is either zero or it is a copy of $M_2(\mathbb{C})$ contained in another subalgebra of M which is $*$-isomorphic to $M_3(\mathbb{C})$.

(ii) M contains a unital subalgebra $*$-isomorphic to one of the following matrix algebras: $M_2(\mathbb{C})$, $M_3(\mathbb{C})$, $M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$.

Statement (i) is summarized and statement (ii) proved in [11].

As a result of the previous theorem it is enough to prove the non-existence of dispersion-free states on algebras $M_3(\mathbb{C})$ and $M_4(\mathbb{C})$.

Definition 28 *A Hidden space of a given quantum system is a set, Ω, with a σ-field, $\mathcal{A}$ of subsets of Ω with the following properties: for each quantum observable A and for each quantum state ρ there is an $\mathcal{A}$-measurable function $f_A : \Omega \mapsto \mathbb{R}$ and a probability measure μ_ρ on $\mathcal{A}$, such that the following conditions are fulfilled:*

(i) For each Borel set $\mathcal{B} \subset \mathbb{R}$ the probability that the value of an observable A is in $\mathcal{B}$ equals $\mu_\rho(f_A^{-1}(\mathcal{B}))$, provided that the system is in the state ρ.

*(ii) (*FUNCTION PRINCIPLE*) If A and B are observables such that $B = g(A)$, where g is a real Borel function, then $f_B = g \circ f_A$.*

Condition (ii) means preserving transformation rules for observables.

Theorem 29 (KOCHEN-SPECKER) *Let $\mathcal{B}(\mathcal{H})$ be an algebra of bounded operators on a separable Hilbert space $\mathcal{H}$ of dimension at least 3. There is no σ-field $(\Omega, \mathcal{A})$ and a map $a \mapsto f_a$ assigning to each self-adjoint element $a \in M$ an $\mathcal{A}$-measurable real function f_a on Ω such that $f_{g(a)} = g \circ f_a$ for any real continuous function g on $\mathbb{R}$.*

In 2004, Döring proved that hidden space does not exist for any von Neumann algebra without a type I_2 and a non-zero abelian direct summand that acts on a separable Hilbert space. This result can be extended to all von Neumann algebras without abelian and a type I_2 direct summand. The following theorem cited from [12] tells us that only the validity of the Function Principle is enough for excluding a hidden space without being necessary to specify the set of states.

Theorem 30 *Let M be a von Neumann algebra without a Type I_2 direct summand and with no non-zero abelian direct summand. There is no σ-field $(\Omega, \mathcal{A})$ and a map $a \mapsto f_a$ assigning to each self-adjoint element $a \in M$ an $\mathcal{A}$-measurable real function f_a on Ω such that $f_{g(a)} = g \circ f_a$ for any real continuous function g on $\mathbb{R}$.*

Proof. We shall prove that existence of such σ-field $(\Omega, \mathcal{A})$ implies existence of a dispersion-free state on $P(M)$ which shall be in contradiction with Theorem 164.

Let $(\Omega, \mathcal{A})$ be the σ-field with the properties stated above. Fix $\omega \in \Omega$. Let us consider the map $s : M_{sa} \mapsto \mathbb{R}$, $s(a) = f_a(\omega)$. If $p \in P(M)$, then $p^2 = p$. Put $m(x) = x^2$, we have that

$$f_p(\omega) = f_{p^2}(\omega) = f_{m(p)}(\omega) = m\big(f_p(\omega)\big) = f_p(\omega)^2 ,$$

hence $s(p) \in \{0, 1\}$.

Let us now take orthogonal non-zero projections $p, q \in M$ and put $x = p + \frac{1}{2}q$.

Let g, h be continuous real functions on $\mathbb{R}$ such that $g(1) = h(\frac{1}{2}) = 1$ and $g(\frac{1}{2}) = h(1) = 0$. Since

$$g(x) = g\big(p + \frac{1}{2}q\big) = g(1)p + g\big(\frac{1}{2}\big)q\,,$$

(the same holds analogously for h) we have that $g(x) = p$ and $h(x) = q$. Setting $u = g + h$ and using the Function Principle we obtain

$$s(p+q) = s\big(g(x)+h(x)\big) = s\big((g+h)(x)\big) = s\big(u(x)\big) = f_{u(x)}(a) = u\big(f_x(\omega)\big) = \\ = g\big(f_x(\omega)\big)+h\big(f_x(\omega)\big) = f_{g(x)}(\omega)+f_{h(x)}(\omega) = s\big(g(x)\big)+s\big(h(x)\big) = s(p)+s(q).$$

Hence s is a finitely-additive dispersion-free measure on $P(M)$. It remains to show that $s(\mathbb{I}) = 1$. It is a consequence of the fact that $s(\mathbb{I}) = f_{\mathbb{I}}(\omega) = 1$ for all $\omega \in \Omega$, where $\mathbb{I} = b(\mathbb{I})$ and b is a constant unit function on $\mathbb{R}$. We have showed that s induces a dispersion-free state on $P(M)$, which is in contradiction with Theorem (26). □

The problem of hidden variables is solved for von Neumann algebras without Type I_2 or non-zero abelian direct summand. It turns out that hidden variables do not exist in these type of algebras. In the case of C^*-algebras we have the following theorem.

Theorem 31 (Hamhalter, (2004)) *Let $\mathcal{A}$ be a simple infinite unital C^*-algebra. Then $\mathcal{A}$ does not admit any dispersion-free quasi-state.*

How does this theory apply to direct measurement of quantum observables? Measurement of quantum systems is always accompanied by an error. Hence it is very strict to demand a dispersion-free state. It is more natural to ask whether there is a hidden space on which the quantum states would have smaller, or even better, arbitrarily small dispersion. The latter is called the problem of *approximate hidden variables* and was introduced by G. W. Mackey in 1968.

Definition 32 *Let ρ be a state on the projection structure $P(\mathcal{A})$ of a C^*-algebra $\mathcal{A}$. The overall dispersion, $\sigma(\rho)$ of ρ is defined by*

$$\sigma(\rho) = \sup\{\rho(p) - [\rho(p)]^2 \mid p \in P(\mathcal{A})\}\,. \tag{4.1}$$

Note 33 *By the previous definition $\sigma(p) \in [0, \frac{1}{4}]$ for all $p \in P(\mathcal{A})$ and $\sigma(p) = 0$ iff $\rho(p) \in \{0, 1\}$, i.e. iff ρ is dispersion-free state.*

Theorem 34 (Hamhalter, (2004)) *Let $\mathcal{A}$ be a unital real rank-zero algebra having no representation onto an abelian C^*-algebra. Then*

$$\sigma(\rho) \geq \frac{2}{9}\,, \tag{4.2}$$

for any state ρ on $\mathcal{A}$.

Theorem (34) provides a global explanation of the nonexistence of hidden variables in quantum theory. The hidden variable is defined as a state on an operator algebra with zero dispersion. First, it was proved by von Neumann that there is no normal dispersion free state on the algebra of all operators on a Hilbert space of dimension at least two. The well known no-go theorem was proved by Plymen to the effect that there is no dispersion free normal state on a von Neumann algebra without central one-dimensional direct summand. Then is was shown that there is no dispersion free state on a von Neumann algebra without abelian part [9].

4.2 Hidden variables in JBW algebras

In this section we seek generalization of the results of hidden variables theory to JBW algebras.

Theorem 35 *Let M be a JBW algebra without associative and Type I_2 direct summand. Then there is no finitely-additive dispersion-free state on $P(M)$.*

Proof. Let M be a JBW algebra with the properties stated above. Suppose that there exists finitely additive dispersion-free measure on $P(M)$. Let us decompose M as follows

$$M = z_1 M \oplus z_2 M \ ,$$

where $z_1 M$ is of type I modular part and $z_2 M$ has no type I modular part. Let ρ be a non-zero $0-1$ state on M.

1) Suppose $\rho(z_2) = 1$. By Proposition (6) there are pairwise orthogonal projections $p_1, \ldots, p_4 \in P(M)$ such that $z_2 = p_1 + \cdots + p_4$ and $p_i \sim_1 p_j$ for all $i, j \in \{1, \ldots, 4\}$. Hence these projections are contained in a subalgebra of isomorphic copy of algebra of matrices $M_4(\mathbb{R})$, which we denote by $\{p_1, \ldots, p_4\} \subseteq M_4(\mathbb{R})$. Since the existence of non-zero $0-1$ state on algebra of real symmetric matrices 4×4 is excluded, we have that $\rho(a) = 0$ for all $a \in M_4(\mathbb{R})$. This property is hereditary to $\{p_1, \ldots, p_4\} \subseteq M_4(\mathbb{R})$, thus ρ must be identically equal zero, i.e. $\rho(z_2) = 0$ which is a contradiction.

2) Now assume that M is of type I. By Theorem (7) M can be uniquely decomposed in the way

$$M = \bigoplus_{n=3}^{\infty} M_n \ ,$$

where M_n is either zero or type I_n, $3 \leq n < \infty$. Now let M_n be a direct summand in the sum above. M_n is either zero or type I_n. If the latter is true then there are n orthogonal abelian projections $q_{j,n} \sim_1 q_{k,n}$ for all $j,k \in \{1,\dots,n\}$ such that for the unit $\mathbb{I}_n \in M_n$ we have that

$$\sum_{j=1}^{n} q_{j,n} = \mathbb{I}_n\,, \quad 3 \leq n < \infty\,.$$

Put $k := [\frac{n}{3}]$ being the whole part of $\frac{n}{3}$; $n = 3k + r$, where $r \in \{0,1,2\}$ satisfies $n \equiv r(mod\,3)$. Now put

$$f_n := \sum_{j=1}^{k} q_{j,n}\,, \quad g_n := \sum_{j=k+1}^{2k} q_{j,n}\,,$$

$$h_n := \sum_{j=2k+1}^{3k} q_{j,n}\,, \quad u_n := \sum_{j=3k+1}^{n} q_{j,n}\,.$$

On employing Proposition (9) we obtain $f_n \sim_1 g_n \sim_1 h_n$ and $u_n \lesssim_1 h_n$. We can write

$$\mathbb{I}_n = f_n + g_n + h_n + u_n\,.$$

Now put

$$f := \sum_{n=3}^{\infty} f_n\,, \quad g := \sum_{n=3}^{\infty} g_n\,,$$

$$h := \sum_{n=3}^{\infty} h_n\,, \quad u := \sum_{n=3}^{\infty} u_n\,.$$

By Proposition (9) $h \sim_1 g \sim_1 f$ and $u \lesssim_1 h$, such that $h + g + f + u = \mathbb{I}$. Now consider the following two possibilities:

(i) $\rho(u) = 0$, $\{h,g,f\} \subseteq M_3(\mathbb{R})$. If $\rho(h+g+f) \neq 0$, then ρ is non-zero on $M_3(\mathbb{R})$, which is a contradiction.

(ii) $\rho(u) = 1$ and $\{h,g,f,u\} \subseteq M_4(\mathbb{R})$, then u is contained in $M_4(\mathbb{R})$ thus $\rho(u) = 0$ which is a contradiction again.

We have showed that if ρ is a $0-1$ state on M, then it is identically zero, proving the statement of the Theorem. □

Theorem 36 *Type I_2 JBW algebras always admit $0-1$ state.*

Proof. Let us denote by U_k a spin factor $U_k = \mathcal{H}_k \oplus \mathbb{R}\mathbb{I}$, where $\mathcal{H}_k$ is the Hilbert space of dimension $k \in \mathbb{N} \cup \{\infty\}$ and $\mathbb{I}$ the identity operator. We shall start the proof by investigating the explicit form of orthogonal projections in U_k.

Let $p \in U_k$ be a projection. Since $U_k = \mathcal{H}_k \oplus \mathbb{R}\mathbb{I}$, there exists a unique pair $(\xi, \lambda\mathbb{I}) \in \mathcal{H}_k \times \mathbb{R}\mathbb{I}$ such that $p = \xi + \lambda\mathbb{I}$. Projection is idempotent, i.e.

$$\xi + \lambda\mathbb{I} = p = p^2 = p \circ p = (\xi + \lambda\mathbb{I}) \circ (\xi + \lambda\mathbb{I}) =$$
$$= 2\lambda\xi + (<\xi, \xi> + \lambda^2)\mathbb{I}\,.$$

Solving the equation above we get

$$\xi(2\lambda - \mathbb{I}) + (<\xi, \xi> + \lambda^2 - \lambda)\mathbb{I} = 0 \Rightarrow \lambda = \frac{1}{2},\ ||\xi|| = \frac{1}{2}\,.$$

The vector $\xi \in \mathcal{H}_k$ such that $||\xi|| = \frac{1}{2}$ can be written in the form $\xi = \frac{1}{2}x$, where $x \in \mathcal{H}_k$ such that $||x|| = 1$. Hence if p and q are two minimal orthogonal projections in U_k with sum $\mathbb{I}$ then

$$p = \frac{1}{2}x + \frac{1}{2}\mathbb{I}\,,$$

$$q = -\frac{1}{2}x + \frac{1}{2}\mathbb{I}\,.$$

Define $\rho : U_k \mapsto \mathbb{R}$ by

$$\rho(\alpha x + \beta\mathbb{I}) = \alpha + \beta\,,$$

where x is the unit vector in $\mathcal{H}_k$ and $\alpha, \beta \in \mathbb{R}$. We will show that ρ is a finitely-additive probability measure on U_k with values in the set $\{0, 1\}$:

(i) $\rho(\mathbb{I}) = \rho(0 + 1 \cdot \mathbb{I}) = 0 + 1 = 1$,

(ii) $\rho(0) = \rho(0 + 0 \cdot \mathbb{I}) = 0 + 0 = 0$,

(iii) Let $n \in \mathbb{N}$ and $\{a_j\}_{j \in \{1,\dots,n\}} \subset U_k$ be set of pairwise orthogonal projections. It is evident that only two of them are non-zero, for instance $a_1 := p$, $a_2 :=$ q and $a_k = 0$ for $k \in \{3, \dots, n\}$. If p, q are non-zero orthogonal projections in U_k then $p + q = \mathbb{I}$. Hence $\rho\big(\sum_{j=1}^n a_j\big) = \rho(p+q) = \rho(\mathbb{I}) = 1 = 1 + 0 + \cdots + 0 = \rho(p) + \rho(q) + \rho(0) + \cdots + \rho(0) = \sum_{j=1}^n \rho(a_j)$.

(iv) Positivity of ρ follows from the fact that ρ takes values only in the set $\{0, 1\}$.

To complete the proof we shall need to extend the results to whole Type I_2 algebra with the help of the fact that Type I_2 algebra is isomorphic to direct sum of $C(X, U_k)$ [see section 6.3 in [14]]. Fix $x \in X$ and define $f, g \in C(X, U_k)$ to be minimal orthogonal projections with sum $\mathbb{I}$, i.e. $f(x) \circ g(x) = (f \circ g)(x) = 0$ and $(f + g)(x) = f(x) + g(x) = \mathbb{I}$. We have that

$$1 = \rho(\mathbb{I}) = \rho((f+g)(x)) = \rho(f(x) + g(x)) = \rho(f(x)) + \rho(g(x))\,,$$

$$\rho(0) = \rho((f \circ g)(x)) = \rho(f(x) \circ g(x)) = 0\,.$$

Thus ρ is a dispersion-free state on $C(X, U_k)$. The proof is complete. $\square$

Corollary 37 *Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be JBW algebra without associative and Type I_2 direct summand. There is no σ-field $(\Omega, \mathcal{D})$ and a map $a \mapsto f_a$ assigning to each self-adjoint element $a \in \mathcal{A}$ a $\mathcal{D}$-measurable real function f_a on Ω such that $f_{g(a)} = g \circ f_a$ for real quadratic polynomial g on $\mathbb{R}$.*

Proof. The proof is analogous to the proof in Theorem (30), i.e. assuming such field exists we find dispersion-free state on $\mathcal{A}$ which is a contradiction. □

Theorem 38 *$M_2(\mathbb{C})$ admits the Function principle.*

Proof. We have already proved that algebra $M_2(\mathbb{C})$ admits $0-1$ state (recall that it is a finitely-additive probability measure with values in the set $\{0, 1\}$, hence it is bounded). Such state extends uniquely to a $0-1$ quasi-functional on $M_2(\mathbb{C})$. Denote by $M_{s.a.}$ a self-adjoint part of $M_2(\mathbb{C})$ and by S the separating set of $0-1$ quasi-functionals on $M_2(\mathbb{C})$ defined on orthogonal pairs. For each $A \in M_{s.a.}$ define map $F_A(\mu) : S \mapsto \mathbb{R}$ by $F_A(\mu) = \mu(A)$. Since $A \in M_{s.a.}$, by virtue of spectral decomposition we have that $A = \lambda_1 p_1 + \lambda_2 p_2$, where p_1, p_2 are orthogonal idempotents with sum $\mathbb{I}$. Let $f \in C(\mathbb{R})$. Matrix function then has the form $f(A) = f(\lambda_1)p_1 + f(\lambda_2)p_2$. Hence

$$F_{f(A)}(\mu) = \mu(f(A)) = \mu(f(\lambda_1)p_1 + f(\lambda_2)p_2) = {}^1 f(\lambda_1)\mu(p_1) + f(\lambda_2)\mu(p_2)\,,$$

$$f \circ F_A(\mu) = f(\mu(A)) = f(\mu(\lambda_1 p_1 + \lambda_2 p_2)) = f(\lambda_1 \mu(p_1) + \lambda_2 \mu(p_2))\,.$$

Since $\mu \in S$ and p_1, p_2 have the properties stated above, it is either

$$\mu(p_1) = 1 \text{ and } \mu(p_2) = 0$$

or vice versa. Without loss of generality we shall assume that the first is true. Then evidently

$$F_{f(A)}(\mu) = f(\lambda_1) = f \circ F_A(\mu)\,,$$

completing the proof. □

Note 39 *Theorem (36) can be viewed as a result of Theorem (38). Indeed, the Function principle implies the existence of non-zero $0-1$ state.*

[1] Recall that quasi-functional is additive with respect to commuting elements. Indeed, p_1, p_2 being orthogonal projections with sum $\mathbb{I}$ commute with each other,

$$p \circ q = q \circ p = pq = qp = 0\,.$$

4.3 Dispersions of states on Jordan algebras

In this section we shall generalize results on overall dispersion of states to Jordan structure. Let A be a Jordan algebra with projective structure $\mathcal{P}(\mathcal{A})$. Overall dispersion of state ϕ is defined by $\sigma(\phi) := \sup\{\phi(p) - \phi(p)^2; p \in P(\mathcal{A})\}$. In our previous work we have shown that JBW algebra with no associative and type I_2 direct summand admits no dispersion-free state. Recently problem of ε-hidden variables has been investigated on von Neumann algebras. The results indicate that lower bound of overall dispersion cannot be improved in general, i.e. $\sigma(\phi) \geq 2/9$. It excludes hypothesis of states possessing an arbitrarily small dispersion. In course of proving similar lemmas and theorems on Jordan structure we have to keep in mind that some type I_n JBW algebras (namely spin factors and $\mathbb{H}_3(\mathbb{O})$) are very different from those of von Neumann algebras. The proof of Theorem (43) has been broken into several lemmas bigger transparency.

Lemma 40 *Let A be a type I_n JBW algebra with $n \geq 2$. Then there are no multiplicative states on A .*

Proof. A is of type I_n then there are n pairwise orthogonal projections such that $p_i \sim_1 p_j$ for all $i, j \in \{1, \ldots, n\}$ summing to $\mathbb{I}$, i.e. $\mathbb{I} = \sum_{j=1}^n p_j$. Now suppose that ϕ is a multiplicative state on A . Recall that $p_1 \sim_1 p_2$ if there is a symmetry $s \in \mathcal{A}$ such that $U_s(p_1) = \{sp_1s\} = p_2$. ϕ is multiplicative, i.e. $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in \mathcal{A}$. From $\phi(ab) = \phi(a)\phi(b) = \phi(b)\phi(a) = \phi(ba)$ we see that ϕ is tracial so $\phi(p_2) = \phi(\{sp_1s\}) = \phi(sp_1s) = \phi(p_1s^2) = \phi(p_1)$. By induction we get that $\phi(p_1) = \phi(p_2) = \cdots = \phi(p_n)$. If $\phi(p_1) = 0$ then $\phi(\mathbb{I}) = \phi\big(\sum_{j=1}^n p_j\big) = \sum_{j=1}^n \phi(p_j) = 0$. If $\phi(p_1) = 1$ then $\phi(\mathbb{I}) = n \geq 2$ which is a contradiction with $\phi(\mathbb{I}) = 1$ completing the proof. $\square$

Lemma 41 *Let A be a JBW algebra admitting a projection p with*

$$p \sim_1 p^\perp. \tag{4.3}$$

Then, for any state φ on A there is a projection q in A such that $\varphi(q) = 1/2$. Hence, $\sigma(\varphi) = 1/4$ for any state φ on A .

Proof. If $p \sim_1 p^\perp$ then the projections p and $p^\perp$ induce a matrix units system and so a unital subalgebra isomorphic to $\mathbb{H}_2(\mathbb{R})$. Since by Lemma (40) there is no multiplicative state on $\mathbb{H}_2(\mathbb{R})$, we can find for a fixed state ϕ a one-dimensional projection e such that $0 < \phi(e) < 1$. Let us assume that $\phi(e) \leq 1/2 \leq \phi(\mathbb{I} - e)$. We will show that there is a continuous path

$\{f(t); t \in [0,1]\}$ connecting e and $\mathbb{I} - e$. Without loss of generality we may assume that

$$e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbb{I} - e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then

$$f(t) = \begin{pmatrix} t & \sqrt{t - t^2} \\ \sqrt{t - t^2} & 1 - t \end{pmatrix}$$

is a path in $\mathcal{P}\big(\mathbb{H}_2(\mathbb{R})\big)$ with $f(0) = e$ and $f(1) = \mathbb{I} - e$. Therefore $\phi\big(f(t_0)\big) = 1/2$ for some $t_0 \in [0,1]$. The proof is complete. □

Lemma 42 *Let A be a JBW algebra without non-zero associative direct summand. Then there is a unital subalgebra A of A such that*

$$A = zA \oplus (\mathbb{I} - z)A, \tag{4.4}$$

where $z \in \mathcal{Z}(A)$, zA is either zero or a copy of $\mathbb{H}_2(\mathbb{R})$ and $(\mathbb{I} - z)A$ is either zero or a copy of $\mathbb{H}_3(\mathbb{R})$.

Proof. It suffices to find a JBW subalgebra of A such that A is of the form

$$\bigoplus_{\alpha \in I} M_\alpha, \tag{4.5}$$

where each M_α is isomorphic either to $\mathbb{H}_2(\mathbb{R})$ or to $\mathbb{H}_3(\mathbb{R})$. Let p be a central projection such that $p\mathcal{A}$ is of finite type I and $(\mathbb{I} - p)\mathcal{A}$ has no nonzero type I finite direct summand. If $\mathbb{I} - p \neq 0$, then by (8) $\mathbb{I} - p$ can be halved and we can find a subalgebra of $(\mathbb{I} - p)\mathcal{A}$ isomorphic to $\mathbb{H}_2(\mathbb{R})$. $p\mathcal{A}$ is either zero or a direct sum of type I_n algebras, where $2 \le n < \infty$ for each n. Each type I_n JBW algebra is isomorphic to $\mathbb{H}_n(R)$ for some algebra R. Since every integer $n \ge 2$ can be written in the form

$$n = 2k + 3l,$$

where k, l are non-negative integers, we can find in each M_n a subalgebra of the form

$$M_{n,1} \oplus M_{n,2},$$

where $M_{n,1}$ is either zero or isomorphic to $\mathbb{H}_2(\mathbb{C})$ and $M_{n,2}$ is by [14], 6.4.1 a direct sum

$$M_1 \oplus M_2 \oplus M_3 \oplus M_4,$$

such that factor representation of each M_j is onto either zero JBW factor or JBW factor isomorphic to $\mathbb{H}(R_j)$, where $R_1 = \mathbb{R}, R_2 = \mathbb{C}, R_3 = \mathbb{H}$ and $R_4 = \mathbb{O}$. Since algebras $\mathbb{C}, \mathbb{H}$ and $\mathbb{O}$ all live in $\mathbb{R}$, the algebras $\mathbb{H}_3(\mathbb{O}), \mathbb{H}_3(\mathbb{H})$ and $\mathbb{H}_3(\mathbb{C})$ contain $\mathbb{H}_3(\mathbb{R})$. Thus, we have found a unital subalgebra of the form (4.5) completing the proof. □

Theorem 43 *Let A be a JBW algebra with no associative direct summand, then*

$$\sigma(\phi) \geq 2/9 \tag{4.6}$$

for any state φ on A .

Proof. By Lemma (42) there is a subalgebra A of A of the form $A = zA \oplus (\mathbb{I} - z)A$ with the properties stated above. Let ϕ be a state on A . By Proposition 2.2 in [9] there is a projection $p \in (\mathbb{I} - z)A$ such that $\phi(p) \in [(1/3)\phi(\mathbb{I} - z), (2/3)\phi(\mathbb{I} - z)]$. By taking projection $q \in zA$ with $\phi(q) = 1/2\phi(z)$ (Lemma (41)) we obtain

$$\frac{1}{3} \leq \phi(p+q) \leq \frac{2}{3}, \tag{4.7}$$

which implies

$$\sigma(\phi) \geq 2/9.$$

□

Theorem (43) says that states with zero dispersion do not exist in JBW algebras, excluding the hidden variables.

Chapter 5

Bell's Inequalities

5.1 General Setting

We shall assume that the possible measurements at site A are described by an order-unit space $(\mathcal{A}, \geq, \mathbb{I})$, abbreviated by $\mathcal{A}$, which is a vector space $\mathcal{A}$ ordered by a convex cone $\mathcal{A}_+ \equiv \{a \in \mathcal{A} \,|\, a \leq 0\}$ with a distinguished element $\mathbb{I} \in \mathcal{A}_+$ whose multiplies eventually dominate every other element of $\mathcal{A}_+$. Preparations correspond to positive, normalized linear functionals on $\mathcal{A}$, called (statistical) states on $\mathcal{A}$. In general setting a measurement with finitely many possible outcomes $i \in I$ is formalized by a finite family $\{a_i\}_{i\in I}$ with $a_i \in \mathcal{A}_+$ and $\sum_{i\in I} a_i = \mathbb{I}$. A preparation is represented by a statistical state ω on $\mathcal{A}$, and $\omega(a_i)$ is the probability for obtaining the result i in an experiment with preparing and measuring devices represented by ω and $\{a_i\}_{i\in I}$, respectively.

A set of correlation experiments is described by a structure called *correlation duality*: a correlation duality consists of two order-unit spaces $\mathcal{A}$ and $\mathcal{B}$ together with a bilinear functional $\widehat{p} : \mathcal{A} \times \mathcal{B} \mapsto \mathbb{R}$ such that $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $a, b \geq 0$ imply $\widehat{p}(a, b) \geq 0$ and $\widehat{p}(\mathbb{I}, \mathbb{I}) = 1$.

The value $\widehat{p}(a_i, b_j)$ represents the probability for obtaining both the result i at site A and result j at site B for measuring devices described by $\{a_i\}_{i\in I} \subset \mathcal{A}_+$ and $\{b_j\}_{j\in J} \subset \mathcal{B}_+$. In C^*-algebraic setting $\mathcal{A}$ and $\mathcal{B}$ are typically self-adjoint parts of elementwise commuting subalgebras of a larger algebra $\mathcal{C}$ and $\widehat{p}$ is given by a state ω on $\mathcal{C}$ by $\widehat{p}(a, b) \equiv \omega(ab)$.

The probability for a certain outcome at B does not depend on measuring devices chosen at A. Indeed, consider two measuring devices $\{a_i\}_{i\in I}$ and $\{a_j^{'}\}_{j\in J}$ at A. Then by definition $\mathbb{I} = \sum_i a_i = \sum_j a_j^{'}$, so that for any $b \in \mathcal{B}$, $\sum_i \widehat{p}(a_i, b) = \sum_j \widehat{p}(a_j^{'}, b) = \widehat{p}(\mathbb{I}, b)$. This assumption in derivation of Bell's inequalities is called *locality* (which is not to be confused with locality in the sense of relativistic causality).

We shall say that (a_1, a_2, b_1, b_2) is *admissible quadruple* if $a_1, a_2 \in \mathcal{A}$, $b_2, b_2 \in \mathcal{B}$ and $-\mathbb{I}_{\mathcal{A}} \leq a_i \leq \mathbb{I}_{\mathcal{A}}$, $i = 1, 2$ and $-\mathbb{I}_{\mathcal{B}} \leq b_j \leq \mathbb{I}_{\mathcal{B}}$, $j = 1, 2$.

Theorem 44 *Let $(\widehat{p}, \mathcal{A}, \mathcal{B})$ be a correlation duality, let $\omega \in \mathcal{A}^*$ be the state $\omega(a) \equiv \widehat{p}(a, \mathbb{I})$, and let (a_1, a_2, b_1, b_2) be an admissible quadruple. Setting*

$$\chi = \frac{1}{2}|\widehat{p}(a_1, b_1 + b_2) + \widehat{p}(a_2, b_1 - b_2)|,$$

one has the following:

(1) $\chi \leq 2$.

(2) *(a) If $\mathcal{A}$ is the Hermitian part of a C^*-algebra, then $\chi \leq \sqrt{2}$.*

(b) If $\chi = \sqrt{2}$ in this case, the following identities hold for all $a \in \mathcal{A}$ and $i = 1, 2$: $\omega([a_i, a]) = 0$, $\omega(a_i^2 a) = \omega(a)$, $\omega\big((a_1 a_2 + a_2 a_1)a\big) = 0$.

(3) If any one of the following condition holds, then $\chi \leq 1$:

(a) $\mathcal{A}$ is classical.

(b) ω is pure on $\mathcal{A}$.

(c) There are states $\xi_\alpha \in \mathcal{A}^*$, $\eta_\alpha \in \mathcal{B}^*$ and positive reals λ_α such that for all $a \in \mathcal{A}$, $b \in \mathcal{B}$, $\widehat{p}(a, b) = \sum \lambda_\alpha \xi_\alpha(a) \eta_\alpha(a)$.

Proof can be found in [21].

By (3c) Bell's inequalities are satisfied even for quantum systems whenever the correlations are produced by mechanism which can be simulated by random generator (producing the *outcome* α with probability λ).

Now let ω be a faithful state on $\mathcal{A}$ defined as in the preceeding theorem. If $\omega(a_i^2 a) = \omega(a)$ for all $a \in \mathcal{A}$, then $\omega\big((a_i^2 - \mathbb{I})a\big) = 0$ for all $a \in \mathcal{A}$ together with faithfulness of ω imply $a_i^2 = \mathbb{I}$. Analogously if $\omega\big((a_1 a_2 + a_2 a_1)a\big) = 0$ for all $a \in \mathcal{A}$ with faithfulness of ω imply $a_1 a_2 + a_2 a_1 = 0$. Hence if $\chi = \sqrt{2}$ when $\mathcal{A}$ is the Hermitian part of a C^*-algebra, the corresponding elements a_1, a_2 and $a_3 = -[a_1, a_2]$ form a realization of the Pauli spin matrices in $\mathcal{A}$. Equation $\omega(a_i^2 a) = \omega(a)$ then implies that the state ω restricted to the 2×2 matrix algebra $M_2(\mathbb{C})$ spanned by $\mathbb{I}, a_1, a_2, a_3$ is the normalized trace.

Since classical, quantum mechanical and quantum field theoretical models are all subsumed in the C^*-algebraic framework, part (2a) informs us that $\chi = \sqrt{2}$ is really the maximal possible correlation. The bound $\chi \leq \sqrt{2}$ constrains "local" quantum theoretical descriptions in the same way that Bell's inequality $\xi \leq 1$ constrains local classical descriptions.

Definition 45 *The* maximal Bell correlation $\beta(\widehat{p},\mathcal{A},\mathcal{B})$ *in a correlation duality* $(\widehat{p},\mathcal{A},\mathcal{B})$ *is*

$$\beta(\widehat{p},\mathcal{A},\mathcal{B}) \equiv \frac{1}{2}\sup|\widehat{p}(a_1,b_1)+\widehat{p}(a_1,b_2)+$$
$$+\widehat{p}(a_2,b_1)-\widehat{p}(a_2,b_2)|,$$

where the supremum is taken over all $a_i \in \mathcal{A}$ *and* $b_j \in \mathcal{B}$ *with* $-\mathbb{I}_{\mathcal{A}} \leq a_i \leq \mathbb{I}_{\mathcal{A}}$ *and* $-\mathbb{I}_{\mathcal{B}} \leq b_i \leq \mathbb{I}_{\mathcal{B}}$.

If $\beta(\widehat{p},\mathcal{A},\mathcal{B}) > 1$ we shall say that Bell's inequalities are violated in $(\widehat{p},\mathcal{A},\mathcal{B})$. In C^*-algebraic setting we shall say that the inequalities are *maximally* violated if $\beta(\widehat{p},\mathcal{A},\mathcal{B}) = \sqrt{2}$.

5.2 Quantum entanglement in von Neumann algebras

Now let us consider a composite system consisting of two subsystems whose observables are given by self-adjoint elements of the von Neumann algebras $M, N \subseteq \mathcal{B}(\mathcal{H})$, respectively. If these two subsystems are in a certain case independent, then the algebras M and N mutually commute, i.e. $M \subset N'$. The algebra of the composite system would be $M \vee N = (M \cup N)''$. A state ϕ on $M \vee N$ is a *product state* if

$$\phi(ab) = \phi(a)\phi(b)\,,$$

for all $a \in M$ and $b \in N$. In many applications of quantum theory the algebra of observables of the composite system may be considered to be of the form $M \otimes N \subset \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H}) \simeq \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$, where M is identified with $M \otimes \mathbb{I}$ and N with $\mathbb{I} \otimes M$. A normal state ϕ on $M \otimes N$ is called *separable*[1] if it is a if it is in the norm closure of the mixture of normal product states. Otherwise ϕ is said to be *entangled*.

We say that von Neumann algebras M, N satisfy *Schlieder property* if for all $a \in M$ and $b \in N$ $ab = 0$ implies that $a = 0$ or $b = 0$.

Definition 46 *Let* $M, N \subset \mathcal{B}(\mathcal{H})$ *be von Neumann algebras such that* $M \subset N'$. *The* maximal Bell correlation *of the pair* (M, N) *in the state* $\phi \in \mathcal{B}(\mathcal{H})^*$ *is*

$$\beta(\phi, M, N) \equiv \sup \frac{1}{2}\phi\big(a_1(b_1+b_2)+a_2(b_1-b_2)\big)\,,$$

where the supremum is taken over all self-adjoint $a_i \in M$ *and* $b_j \in N$, $i, j = 1, 2$, *with norm less than or equal to* 1.

[1]also termed decomposable, classically correlated or unentangled by many authors

Proposition 47 *Every state on $M \otimes N$ is separable if and only if either M or N is abelian.*

Proof can be found in [19].
This implies that if both systems are quantum, i.e. both algebras are non-commutative, then there exist entangled states of the composite system.

Proposition 48 *Let $M, N \subset \mathcal{B}(\mathcal{H})$ be mutually commuting von Neumann algebras. If either M or N is abelian, then $\beta(\phi, M, N) = 1$ for all states $\phi \in \mathcal{B}(\mathcal{H})^*$.*

Proof can be found in [21].

Proposition 49 *If $M, N \subset \mathcal{B}(\mathcal{H})$ are nonabelian, mutually commuting von Neumann algebras satisfying the Schlieder property, then there exists a normal state $\phi \in \mathcal{B}(\mathcal{H})^*$ such that $\beta(\phi, M, N) = \sqrt{2}$.*

Proof can be found in [15].

5.3 Maximal Violation

In JBW algebraic setting we have the following modification: we say that JBW algebras $\mathcal{A}$ and $\mathcal{B}$ satisfy the Schlieder property if $0 = a \circ b$ for $a \in \mathcal{A}$ and $b \in \mathcal{B}$ implies either $a = 0$ or $b = 0$.

Theorem 50 *Let $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be operator-commuting JB subalgebras of a JB algebra $\mathcal{A}$. The following conditions are equivalent:*

(i) If $a_i \in \mathcal{A}_i$, $i = 1, \ldots, n$ are nonzero elements, then the product $a_1 \circ \cdots \circ a_n$ is a nonzero element in $\mathcal{A}$.

(ii) Any n-tuple of states $\varphi_1 \in \mathcal{A}_1^*, \varphi_2 \in \mathcal{A}_2^*, \ldots, \varphi_n \in \mathcal{A}_n^*$ has a common extention to $\mathcal{A}$.

Proof can be found in [13].

Theorem 51 *Let $\mathcal{A}$ and $\mathcal{B}$ be mutually operator-commuting non-associative JBW algebras in a JBW algebra $\mathcal{C}$ satisfying the Schlieder property. Then there is a state φ on $\mathcal{C}$ and self-adjoint contractions $a_1, a_2 \in \mathcal{A}$, $b_1, b_2 \in \mathcal{B}$, such that*

$$\left|\varphi\Big(\frac{1}{2}\big(a_1 \circ (b_1 + b_2) + a_2 \circ (b_1 - b_2)\big)\Big)\right| = \sqrt{2}.$$

Proof. Since $\mathcal{A}$ and $\mathcal{B}$ are non-associative, by structural theory (see 5.2 in [14]) they both contain a copy of $\mathbb{H}_2(\mathbb{R})$. Let us consider an algebra $\mathbb{H}_2(\mathbb{R})$ generated by unit and σ_1 and σ_2, the Pauli spin matrices (non-trivial symmetries).

By Theorem (50) states $\varphi_{1,2}$ on $\mathcal{A}$, $\mathcal{B}$ respectively, have a common extension φ to $\mathcal{C}$. In other words, $\mathcal{A}$ and $\mathcal{B}$ generate subalgebra isomorphic to $\mathbb{H}_2(\mathbb{R}) \otimes \mathbb{H}_2(\mathbb{R}) = \mathbb{H}_4(\mathbb{R})$.

Let us take projections

$$p = 1/2(\mathbb{I} + \sigma_1)\,,$$

$$q = 1/2(\mathbb{I} + \sigma_2)\,.$$

Then $||pq - qp|| = 1/2$. In a similar way we can choose projections $r, s \in \mathcal{B}$ with $||rs - sr|| = 1/2$. Now set $a_1 = 2p - \mathbb{I}$, $a_2 = 2q - \mathbb{I}$, $b_1 = 2r - \mathbb{I}$, $b_2 = 2s - \mathbb{I}$ and $z = a_1 \otimes (b_1 + b_2) + a_2 \otimes (b_1 - b_2)$. Then

$$z^2 = 4\mathbb{I} \otimes \mathbb{I} + 16[p, q] \otimes [s, r].$$

By the tensor product property, $||[p, q] \otimes [s, r]|| = 1/4$ and $\|z^2\| = 4 + 4 = 8$. Therefore, we can find a norm-one functional ψ on $\mathcal{C}$ such that $\psi([p, q] \otimes [s, r]) = 1/4$. Consequently,

$$\psi(z^2) = 4 + 16\frac{1}{4} = 8\,,$$

hence $8 = |\psi(z^2)| \leq ||z||^2$, yielding $||z|| \geq 2\sqrt{2}$. Again, we can find a normal state φ such that $|\varphi(z)| = ||z|| \geq 2\sqrt{2}$. Moreover, $|\varphi(z)|^2 \leq |\varphi(z^2)| = 8$. Then $2\sqrt{2} \leq |\varphi(z)| \leq 2\sqrt{2}$, hence $|\varphi(z)| = 2\sqrt{2}$ and $|\varphi(\frac{1}{2}z)| = \sqrt{2}$. The proof is complete. □

In this section we shall generalize the Bell's inequalities for more general correlation duality. Let X denote normed space, $Q(\cdot, \cdot)$ a sesquilinear positive form on X with $||Q|| = 1$. Let $||x||_Q \equiv Q(x, x)^{1/2}$ be a seminorm on X and

$$B_Q \equiv \sup_{x,y,a,b \in X_1} \frac{1}{2}|Q(x, a + b) + Q(y, a - b)|\,,$$

where X_1 is a unit ball in X.

Lemma 52 *In any indefinite inner product space*

$$||u + v|| + ||u - v|| \leq 2\sqrt{2}\,,$$

whenever $||u||$, $||v|| \leq 1$.

Proof.

$$||u + v|| + ||u - v|| =$$

$$= \sqrt{||u||^2 + ||v||^2 + 2Re < u, v >} + \sqrt{||u||^2 + ||v||^2 - 2Re < u, v >} \leq$$

$$\leq \sqrt{2+2Re < u,v >} + \sqrt{2-2Re < u,v >}\,.$$

Set $t = Re < u,v >$. It is clear that $t \in [-1,1]$. Consider function $\omega(t) = \sqrt{2+2t}+\sqrt{2-2t}$ on $[-1,1]$. ω is even and decreasing on $[0,1]$. The following estimation holds $2 \leq \omega(t) \leq 2\sqrt{2}$ on $[0,1]$. Hence $\|u+v\| + \|u-v\| \leq 2\sqrt{2}\,.$ $\square$

Theorem 53 *With all the notation and conditions above, $B_Q \leq \sqrt{2}$.*

Proof. On employing the Schwarz inequality we obtain

$$|Q(x,a+b)| \leq Q(x,x)^{1/2}Q(a+b,a+b)^{1/2} \leq ||x||_Q \cdot ||a+b||_Q \leq ||a+b||_Q\,.$$

Similarly

$$|Q(y,a-b)| \leq ||a-b||_Q\,.$$

As $||a||_Q$, $||b||_Q \leq 1$ by the previous Lemma we have that

$$B_Q \leq \frac{1}{2}\big(||a+b||_Q + ||a-b||_Q\big) \leq \frac{1}{2}2\sqrt{2} = \sqrt{2}\,.$$

$\square$

Let us now consider the following example of saturation. Let $X = \mathcal{B}(\mathcal{H})$, $\xi \in \mathcal{H}$ with $||\xi|| = 1$ and $Q(x,y) =< x\xi, y\xi >$. Fix symmetries $a, b \in X$ such that $||a\xi|| = ||b\xi|| = 1$ and $< a\xi, b\xi >= 0$.

Take symmetries $x, y \in \mathcal{B}(\mathcal{H})$ with

$$x\xi = \frac{(a+b)\xi}{||(a+b)\xi||}$$

$$y\xi = \frac{(a-b)\xi}{||(a-b)\xi||}\,.$$

Then

$$\frac{1}{2}|Q(x,a+b) + Q(y,a-b)| = \frac{1}{2}\Big|\big\langle x\xi, (a+b)\xi > + < y\xi, (a-b)\xi\big\rangle\Big| =$$

$$= \frac{1}{2}\Big|\big\langle \frac{(a+b)\xi}{\|(a+b)\xi\|}, (a+b)\xi\big\rangle + \big\langle \frac{(a-b)\xi}{\|(a-b)\xi\|}, (a-b)\xi\big\rangle\Big| =$$

$$= \frac{1}{2}\big(\|a\xi + b\xi\| + \|a\xi - b\xi\|\big) = \frac{1}{2}2\sqrt{2} = \sqrt{2}\,,$$

hence $\sqrt{2}$ is the best estimation.

Chapter 6

Conclusion

In the section 4.3 we succeeded to prove that in JBW algebras with no nonzero associative direct summand there is no dispersion-free state, excluding the Hidden variables theory. Moreover, the lower bound for overall dispersion cannot be improved, excluding hypothesis of states carrying an arbitrarily small dispersion.

In the final section 5.3 we proved that in mutually operator-commuting non-associative JBW algebras satisfying Schlieder property, there exist state and self-adjoint contractions such that Bell's inequality is maximally violated. Next we generalized Bell's inequality for more general correlation duality concluding that the bound $\sqrt{2}$ is the best estimation.

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Prohlášení

Prohlašuji, že jsem svou diplomovou práci vypracoval samostatně a použil jsem pouze podklady (literaturu, projekty, SW atd.) uvedené v přiloženém seznamu.

Nemám závažný důvod proti užití tohoto školního díla ve smyslu § 60 Zákona č.121/2000 Sb., o právu autorském, o právech souvisejících s právem autorským a o změně některých zákonů (autorský zákon).

V Praze dne .... Podpis ....