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Three-body Interactions in Mean-Field Model of Nuclei and Hypernuclei

Bc. Jan Pokorný

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Supervisor: Mgr. Petr Veselý, Ph.D.

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DIPLOMOVÁ PRÁCE

Tříčásticové interakce v modelu středního pole jader a hyperjader

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Školitel: Mgr. Petr Veselý, Ph.D.

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Thesis title: **Three-body Interactions in Mean-Field Model of Nuclei and Hypernuclei**

Author: Jan Pokorný

Department: Department of Physics FNSPE CTU in Prague
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Supervisor: Mgr. Petr Veselý, Ph.D.

Abstract: In the thesis, we study energy spectra of medium-mass hypernuclei. We develop two computational methods based on the mean-field approximation — Hartree-Fock method in the proton-neutron-Λ (p-n-Λ) formalism, and the NΛ Tamm-Dancoff Approximation. Force acting among nucleons is represented by the chiral N^2LO_{sat} potential. Force between Λ particles and nucleons is represented by the chiral LO YN potential. The main focus is put on derivation of these methods using a Hamiltonian with explicit three-body NNN and ΛNN potentials. We performed calculations of hypernuclei ${}^{17}_{\Lambda}$O, ${}^{41}_{\Lambda}$Ca, ${}^{49}_{\Lambda}$Ca, and ${}^{16}_{\Lambda}$O, ${}^{40}_{\Lambda}$Ca, ${}^{48}_{\Lambda}$Ca using the HF method in the p-n-Λ formalism, and the NΛ TDA method. We study the effect of the three-nucleon NNN force within the N^2LO_{sat} potential on the description of energy spectra of the mentioned hypernuclei.

Keywords: hypernuclei, NNN interactions, Hartree-Fock method, Tamm-Dancoff Approximation, mean-field model

Název práce: **Tříčásticové síly v modelu středního pole jader a hyperjader**

Autor: Jan Pokorný

Katedra: Katedra Fyziky FJFI ČVUT v Praze
Obor studia: Experimentální jaderná a částicová fyzika
Druh práce: Diplomová práce

Školitel: Mgr. Petr Veselý, Ph.D.

Abstrakt: V této práci se věnujeme studiu energetických spekter středně těžkých hyperjader. Vyvíjíme dvě metody založené na aproximaci středního pole — Hartree-Fockovu metodu v proton-neutron-Λ (p-n-Λ) formalismu a NΛ Tamm-Dancoffovu aproximaci. Síla působící mezi nukleony je popsána chirální interakcí $\mathrm{N^2LO_{sat}}$. Síla mezi částicemi Λ a nukleony je reprezentována chirálním LO YN potenciálem. Ústřední téma práce je odvození zmíněných metod pomocí Hamiltoniánu s explicitními tříčásticovými NNN a ΛNN interakcemi. Provedli jsme výpočty hyperjader ${}^{17}_{\Lambda}\mathrm{O}$, ${}^{41}_{\Lambda}\mathrm{Ca}$, ${}^{49}_{\Lambda}\mathrm{Ca}$ a ${}^{16}_{\Lambda}\mathrm{O}$, ${}^{40}_{\Lambda}\mathrm{Ca}$, ${}^{48}_{\Lambda}\mathrm{Ca}$ s použitím HF metody v p-n-Λ formalismu a NΛ TDA metody. Studujeme efekt třínukleonové NNN síly potenciálu $\mathrm{N^2LO_{sat}}$ na popis energetických spekter zmíněných hyperjader.

Klíčová slova: hyperjádra, NNN interakce, Hartree-Fockova metoda, Tamm-Dancoffova aproximace, model středního pole

Contents

Chapter 1

Introduction

The theory of nuclear structure aims to study the properties of atomic nuclei as bound systems of nucleons. Most microscopic models of nuclear structure are rooted in the non-relativistic quantum mechanics [1]. It is assumed that nucleons are "point-like" particles despite general empirical evidence of their inner quark-gluon structure. The quark and gluonic degrees of freedom, as well as non-nucleonic degrees of freedom (mesons, Δ isobars), are considered usually for construction of the interactions among nucleons. One issue of the nuclear structure theory is construction of the force acting among nucleons. Other issue is application of this force in quantum many-body methods.

Several theoretical models of nucleon-nucleon (NN) potentials have been developed. The first class of the NN potentials is of phenomenological type. These effective forces rely on a set of parameters which are usually fitted directly to the bulk properties of certain set of doubly-magic nuclei [2]. Examples of the effective potentials are the Skyrme [3] and Gogny [4] forces. These interactions are traditionally used in self-consistent mean-field models. Alternatively, self-consistent mean-field models of nuclear structure are constructed from the effective energy-density functionals without any reference to a force among nucleons [2].

The NN potentials of the second class are derived from the microscopic theory of the NN scattering and reproduce the experimental NN scattering data. These potentials are often called realistic. They are either constructed from meson exchanges, such is the case of Nijmegen [5], Argonne V18 [6], and CD-Bonn [7], or are derived from the chiral effective field theory which satisfies all symmetries of QCD with nucleons and pions as the degrees of freedom (ChPT potentials) [8]. ChPT allows to improve step by step the precision of the NN interactions by going into the higher orders of perturbation. Starting from the next-to-next-to leading order (N^2LO), the three-body NNN interactions appear. A detailed discussion of

the ChPT can be found in Refs. [9, 10]. Using the bare realistic NN potentials as the input for various nuclear models often brings practical computational challenges which are caused by the slow convergence and the presence of the strong repulsive core in these forces. Typically, renormalization methods, such as V_{low-k} [11], SRG [12], UCOM [13], and G-Matrix [14] are introduced to address these difficulties. The G-Matrix has a physical interpretation of an effective potential which describes the NN interaction inside the nuclear medium [15].

Various models, which use selected NN potentials, describing the binding energies, charge radii, the excitation spectra, and other properties nuclei, have been introduced. Generally, light and medium-mass nuclei can be described by *ab initio* methods. These models describe the nuclear structure with minimum of approximations. However, their computational complexity rapidly increases with A. Examples of *ab initio* methods are No-Core Shell Model [16], Coupled Cluster Model [17], Fermionic Molecular Dynamics [18], Self-Consistent Green's Function Method [19], and Green's Function Monte Carlo Method [20].

Applicability of self-consistent mean-field models, such as the Hartree-Fock (HF) or Hartree-Fock-Bogoliubov (HFB) methods [21], ranges from ^{16}O to the heaviest nuclei. The HF method is used for description of closed-shell nuclei and the HFB method is applied on the open-shell nuclei. The HF calculations starting from realistic potentials yield single-particle spectra with too big gaps between the major shells as well as compressed nuclear radii. These problems have been shown in HF calculations with the Argonne V18 + UCOM interaction [22] and independently in a calculations based on the CD-Bonn + V_{low-k} interaction [23]. In both studies, the realistic NN potential was supplemented by phenomenological density-dependent (DD) term which simulates the repulsive effect of three-body NNN interaction. The DD term was found to be of crucial importance for generating the single-particle spectra in good agreement with the empirical ones. The self-consistent mean field generated by the HF method with the realistic NN interactions corrected by the DD term was used as a starting point for the beyond mean-field calculations of several medium and heavy nuclei within the Equation of Motion Phonon Method (EMPM) [24, 25, 26]. The EMPM solves a set of equations of motion to generate a multi-phonon basis that can be used for diagonalizing the nuclear Hamiltonian. The many-body correlations in the nuclear eigenstates obtained in the EMPM calculations are crucial for the description of the nuclear ground-state properties [27] as well as the properties of the excited states [28, 29, 30, 31]. The importance of the DD term in the aforementioned calculations calls for the direct implementation of the three-body NNN force which naturally occurs in the potentials derived from ChPT.

Throughout this work, the chiral NN + NNN potential N^2LO_{sat} [32] is employed. The parameters of the N^2LO_{sat} potential were fitted not only to reproduce experimental NN scattering data but also other properties of light nuclear systems [32]. This potential also gives good saturation properties of nuclear matter [32].

The theory of nuclear structure can be extended to study exotic nuclear systems, namely hypernuclei. A hypernucleus is a bound nuclear system, in which a hyperon with non-zero strangeness $(\Lambda, \Sigma, \Xi, \Omega)$ is present. With the exception of the Σ^0, the hyperons decay predominantly weakly which results in their rather long lifetime $\approx 10^{-10}$ s. The first hypernucleus has been observed in 1952 by J. Pniewski and M. Danysz [33]. To this day, approximately 30 species of the Λ hypernuclei have been discovered starting from the lightest hypernucleus ${}^{3}_{\Lambda}$H up to the heaviest hypernuclear systems ${}^{208}_{\Lambda}$Pb and ${}^{208}_{\Lambda}$Bi. Experimental studies of hypernuclei have been performed by many laboratories worldwide (CERN, BNL, KEK, FINUDA, JLab, JPARC, GSI, MAMI-C) [34, 35, 36].

Several hyperon-nucleon (YN) potentials – both bare and effective – have been developed, and used in hypernuclear models. Examples of these potentials are the Nijmegen model ESC08 [37], Jülich meson exchange model [38], and the chiral YN interaction at the LO [39] and recently at the NLO [40].

In Ref. [41], the chiral two-body potential N^2LO_{opt} which minimizes the effect of the three-body force [42], corrected by density-dependent (DD) term which simulates the repulsive effect of NNN force, has been used. The ΛN force was described by the effective G-matrix derived from the Nijmegen interaction within the parametrization ESC08c [43]. This force is strongly dependent on the Fermi momentum k_F. In addition, the symmetric spin-orbit (SLS) and anti-symmetric spin-orbit (ALS) parts of the potential are not treated explicitly but only within the Scheerbaum Approximation [44]. This would not allow us to study the dependence of hypernuclear spectra on the spin part of the ΛN potential beyond the approximation of Hartree-Fock method. In this thesis, we employ the chiral LO YN potential [39] for calculations of hypernuclei.

The main focus of our research is the study of the spectra of medium-mass and heavy single-Λ hypernuclei. Our long-term goal is to provide an *ab initio* description

of structure of these hypernuclei. The thesis provides first steps towards this goal. We implement the Hamiltonian with realistic baryon chiral forces, which include explicitly the chiral three-body NNN force, and develop two computational methods based on the mean-field approximation - the Hartree-Fock (HF) method in the proton-neutron-Λ (p-n-Λ) formalism and the nucleon-Λ TDA (NΛ TDA) method.

This thesis is organized as follows: The derivation of the Hartree-Fock method in the p-n-Λ formalism is shown in Chapter 2. In Chapter 3, coupling of the Λ to the nuclear core is described within the NΛ TDA method. The implementation of the three-body NNN force is described in Chapter 4. We discuss our results in Chapter 5. We summarize our conclusions in Chapter 6. Technical details are presented in two Appendices.

Chapter 2

Hypernuclear mean-field model with three-body interactions

We describe the single-Λ hypernucleus as a many-body system consisting of nucleons and one Λ hyperon. Properties of the hypernucleus are determined by the Hamiltonian

$$\widehat{H} = \widehat{T}^N + \widehat{T}^\Lambda + \widehat{V}^{NN} + \widehat{V}^{NNN} + \widehat{V}^{\Lambda N} + \widehat{V}^{\Lambda NN} - \widehat{T}_{CM}. \tag{2.1}$$

Here, $\widehat{T}^N$ and $\widehat{T}^\Lambda$ denote kinetic energy operators of nucleons and the Λ particle, respectively. The terms $\widehat{V}^{NN}$ and $\widehat{V}^{\Lambda N}$ stand for the two-body NN and ΛN potentials. The three-body interactions are included in the terms $\widehat{V}^{NNN}$ and $\widehat{V}^{\Lambda NN}$. The term $\widehat{T}_{CM}$ denotes the center-of-mass kinetic operator

$$\widehat{T}_{CM} = \frac{1}{2[(A-1)M + M_\Lambda]} \left(\sum_{a=1}^{A} \widehat{\vec{P}}_a^{\,2} + 2 \sum_{a<b} \widehat{\vec{P}}_a \cdot \widehat{\vec{P}}_b \right), \tag{2.2}$$

where $M \approx 938$ MeV is the nucleon mass, $M_\Lambda \approx 1116$ MeV is the mass of the Λ hyperon, A is the baryon number, and $\widehat{\vec{P}}_a$ is the momentum operator of the a-th particle.

The hypernuclear mean field is constructed self-consistently by the Hartree-Fock (HF) method. In Section 2.1, we derive the HF method for a system of identical fermions with two-body interactions in the formalism of the second quantization. In Section 2.2, we derive the HF method in the proton-neutron-Λ formalism with three-body NNN and ΛNN interactions.

2.1 Hartree-Fock method in the second quantization

In this section we derive Hartree-Fock equations in the formalism of second quantization, i.e. in terms of creation (annihilation) operators, $\alpha^\dagger(\alpha)$.

Let us consider a system of A identical fermions interacting through two-body potential $\widehat{V}(\vec{r}_1, \vec{r}_2)$. The Hamiltonian of this system in the second quantization is given by

$$\widehat{H} = \sum_{ij} t_{ij} \alpha_i^\dagger \alpha_j + \frac{1}{4} \sum_{ijkl} V_{ijkl} \alpha_i^\dagger \alpha_j^\dagger \alpha_l \alpha_k, \tag{2.3}$$

where $\alpha_i^\dagger$ creates the single-particle state $|i\rangle$ ($\alpha_i^\dagger|0\rangle = |i\rangle$) and α_i annihilates the single-particle state $|i\rangle$ ($\alpha|i\rangle = |0\rangle$). The ket $|0\rangle$ denotes the particle vacuum. In Eq. (2.3), the matrix elements of the kinetic energy operator read $t_{ij} = \langle i|\widehat{T}|j\rangle$ and the matrix elements of the potential $\widehat{V}(\vec{r}_1, \vec{r}_2)$ are antisymmetrized

$$V_{ijkl} = \langle ij|\widehat{V}(\vec{r}_1, \vec{r}_2)|kl\rangle - \langle ij|\widehat{V}(\vec{r}_1, \vec{r}_2)|lk\rangle = \langle ij|\widehat{V}(\vec{r}_1, \vec{r}_2)|kl - lk\rangle. \tag{2.4}$$

The antisymmetrized ground-state wave function of the studied system is a Slater determinant

$$|\Psi_0\rangle = \prod_{i=1}^{A} \alpha_i^\dagger |0\rangle, \tag{2.5}$$

where the product in Eq. (2.5) runs over the lowest single-particle states.

Throughout our work, we express all physical states in the basis of izotropic harmonic oscillator. The creation (annihilation) operators $\alpha_i^\dagger(\alpha_i)$ correspond to the single-particle states $|i\rangle$. Kets $|i\rangle$ can be expanded into another basis spanned by the operators $\alpha_i'^\dagger(\alpha_i')$. States $|i\rangle$ and $|i'\rangle$ from both bases and their corresponding creation (annihilation) operators are connected through a unitary transformation U

$$|i'\rangle = \sum_j U_{ij} |j\rangle, \tag{2.6a}$$

$$\alpha_i'^\dagger = \sum_{ij} U_{ij} \alpha_j^\dagger, \tag{2.6b}$$

$$\alpha_i' = \sum_{ij} \alpha_j U_{ij}^* = \sum_{ij} U_{ji}^\dagger a_j. \tag{2.6c}$$

The inverse transformation reads

$$|i\rangle = \sum_j U^*_{ji}|j'\rangle, \tag{2.7a}$$

$$\alpha^\dagger_i = \sum_{ij} U^\dagger_{ij}\alpha'^\dagger_j = \sum_{ij} \alpha'^\dagger_j U^*_{ji}, \tag{2.7b}$$

$$\alpha_i = \sum_{ij} U_{ji}\alpha'_j. \tag{2.7c}$$

The basis spanned by the operators $\alpha'^\dagger_i(\alpha'_i)$ is called the self-consistent basis. The ground state expressed in the self-consistent basis is defined as the Hartree-Fock state $|\mathrm{HF}\rangle$

$$|\mathrm{HF}\rangle = \prod_{i=1}^{A} \alpha'^\dagger_i|0\rangle. \tag{2.8}$$

It is convenient to define the density matrix of the HF state ρ^{HF}:

$$\rho^{\mathrm{HF}}_{ji} = \langle \mathrm{HF}|\alpha^\dagger_i\alpha_j|\mathrm{HF}\rangle. \tag{2.9}$$

Using the transformations (2.7b) and (2.7c) leads to the following relation

$$\rho^{\mathrm{HF}}_{ji} = \sum_{kl} U^*_{ki}U^T_{jl}\langle \mathrm{HF}|\alpha'^\dagger_k\alpha'_l|\mathrm{HF}\rangle, \tag{2.10}$$

where we applied the following identity

$$\begin{aligned}\langle \mathrm{HF}|\alpha'^\dagger_k\alpha'_l|\mathrm{HF}\rangle &= \langle \mathrm{HF}|\{\alpha'^\dagger_k,\alpha'_l\}|\mathrm{HF}\rangle - \langle \mathrm{HF}|\alpha'_l\alpha'^\dagger_k|\mathrm{HF}\rangle \\ &= \delta_{kl}\langle \mathrm{HF}|\mathrm{HF}\rangle - \langle \mathrm{HF}|\alpha'_l\alpha'^\dagger_k|\mathrm{HF}\rangle = \delta_{kl}, \quad \forall k : \varepsilon_k \le \varepsilon_F,\end{aligned} \tag{2.11}$$

where ε_k is the energy of the k-th level and ε_F is the energy of the highest occupied level (the Fermi level). We denote the occupied state k as $k-\mathrm{occ.}$. For all occupied single-particle states k, Eq. (2.10) gives

$$\rho^{\mathrm{HF}}_{ji} = \sum_{k-\mathrm{occ.}} U^*_{ki}U^T_{jk} = \sum_{k-\mathrm{occ.}} U^T_{jk}U^*_{ki} = (U^TU^*)_{ji}. \tag{2.12}$$

The next step of the derivation is the construction of the energy functional

$$\langle \mathrm{HF}|\widehat{H}|\mathrm{HF}\rangle = \sum_{ij} t_{ij}\langle \mathrm{HF}|\alpha^\dagger_i\alpha_j|\mathrm{HF}\rangle + \frac{1}{4}\sum_{ijkl} V_{ijkl}\langle \mathrm{HF}|\alpha^\dagger_i\alpha^\dagger_j\alpha_l\alpha_k|\mathrm{HF}\rangle. \tag{2.13}$$

The second term in Eq. (2.13) together with the transformations (2.7b) and (2.7c)

gives

$$\begin{aligned}\langle \mathrm{HF}|\alpha_i^\dagger\alpha_j^\dagger\alpha_l\alpha_k|\mathrm{HF}\rangle &= \sum_{opqr} U^*_{oi}U^*_{pj}U^T_{lr}U^T_{kq}\langle \mathrm{HF}|\alpha_o'^\dagger\alpha_p'^\dagger\alpha'_r\alpha'_q|\mathrm{HF}\rangle \\ &= \sum_{opqr-\mathrm{occ.}} U^*_{oi}U^*_{pj}U^T_{lr}U^T_{kq}(\delta_{oq}\delta_{pr}-\delta_{or}\delta_{pq}) \\ &= \sum_{op-\mathrm{occ.}} (U^*_{oi}U^T_{ko}U^*_{pj}U^T_{lp}-U^*_{oi}U^T_{lo}U^*_{pj}U^T_{kp}) \\ &= (U^TU^*)_{ki}(U^TU^*)_{lj}-(U^TU^*)_{li}(U^TU^*)_{kj}, \end{aligned} \tag{2.14}$$

where we applied the following identity analogous to Eq. (2.11)

$$\langle \mathrm{HF}|\alpha_o'^\dagger\alpha_p'^\dagger\alpha'_r\alpha'_q|\mathrm{HF}\rangle = \delta_{oq}\delta_{pr}-\delta_{or}\delta_{pq}, \quad \forall o,p:\varepsilon_o,\varepsilon_p\le\varepsilon_F. \tag{2.15}$$

The energy functional (2.13) therefore reads

$$\begin{aligned}\langle \mathrm{HF}|\widehat{H}|\mathrm{HF}\rangle &= \sum_{ij}\sum_{o-\mathrm{occ.}} t_{ij}U^T_{jo}U^*_{oi} \\ &\quad + \frac{1}{4}\sum_{ijkl}\sum_{op-\mathrm{occ.}} V_{ijkl}U^T_{ko}U^*_{oi}U^T_{lp}U^*_{pj} - \frac{1}{4}\sum_{ijkl}\sum_{op-\mathrm{occ.}} V_{ijkl}U^T_{lo}U^*_{oi}U^T_{kp}U^*_{pj} \\ &= \sum_{ij} t_{ij}(U^TU^*)_{ji} + \frac{1}{4}\sum_{ijkl} V_{ijkl}(U^TU^*)_{ki}\left(U^TU^*\right)_{lj} \\ &\quad - \frac{1}{4}\sum_{ijkl} V_{ijkl}(U^TU^*)_{li}(U^TU^*)_{kj}. \end{aligned} \tag{2.16}$$

We minimize the functional (2.16) with respect to the variations of the transformation U

$$\delta\langle \mathrm{HF}|\widehat{H}|\mathrm{HF}\rangle = \frac{\delta\langle \mathrm{HF}|\widehat{H}|\mathrm{HF}\rangle}{\delta U}\delta U + \frac{\delta\langle \mathrm{HF}|\widehat{H}|\mathrm{HF}\rangle}{\delta U^*}\delta U^* = 0. \tag{2.17}$$

Here, U is unitary, hence we get two equivalent conditions

$$\frac{\delta\langle \mathrm{HF}|\widehat{H}|\mathrm{HF}\rangle}{\delta U} = 0, \tag{2.18a}$$

$$\frac{\delta\langle \mathrm{HF}|\widehat{H}|\mathrm{HF}\rangle}{\delta U^*} = 0. \tag{2.18b}$$

Variation of the functional in Eq. (2.16) with respect to U^* gives

$$\begin{aligned}\frac{\delta\langle \mathrm{HF}|\widehat{H}|\mathrm{HF}\rangle}{\delta U^*_{op}} &= \sum_j t_{pj}U^T_{jo} + \frac{1}{4}\sum_{jkl} V_{pjkl}(U^TU^*)_{lj}U_{ko} \\ &\quad + \frac{1}{4}\sum_{ikl} V_{ipkl}(U^TU^*)_{ki}U_{lo} \\ &\quad - \frac{1}{4}\sum_{jkl} V_{pjkl}(U^TU^*)_{kj}U_{lo} \\ &\quad - \frac{1}{4}\sum_{ikl} V_{ipkl}(U^TU^*)_{li}U_{ko}. \end{aligned} \tag{2.19}$$

Due to hermiticity and antisymmetry, the matrix elements V_{ijkl} satisfy the following identities

$$V_{ijkl} = -V_{jikl} = -V_{ijlk} = V_{jilk}, \tag{2.20a}$$

$$V_{ijkl} = V_{klij}. \tag{2.20b}$$

Applying (2.20a) and (2.20b) on (2.19) yields the equation

$$\frac{\delta\langle \mathrm{HF}|\widehat{H}|\mathrm{HF}\rangle}{\delta U^*_{op}} = \sum_j \left[t_{pj} + \sum_{kl} V_{pkjl}(U^T U^*)_{lk} \right] U^T_{jo} = 0. \tag{2.21}$$

The unitarity of the matrix U presents following restriction

$$(U^T U^*)_{op} - \mathbb{I}_{op} = 0. \tag{2.22}$$

The variational problem with the restriction (2.22) is expressed as

$$\begin{aligned} &\frac{\delta}{\delta U^*_{op}} \left[\langle \mathrm{HF}|\widehat{H}|\mathrm{HF}\rangle - \varepsilon \left(U^T U^* - \mathbb{I} \right) \right] \\ &= \sum_j \left[t_{pj} + \sum_{kl} V_{pkjl}(U^T U^*)_{lk} \right] U^T_{jo} - \sum_j \varepsilon_p \mathbb{I}_{pj} U^T_{jo} = 0. \end{aligned} \tag{2.23}$$

By substituting the density matrix identity (2.12) we obtain the equation

$$\sum_j \left(t_{pj} + \sum_{kl} V_{pkjl} \rho^{\mathrm{HF}}_{lk} \right) U^T_{jo} = \sum_j \varepsilon_p \mathbb{I}_{pj} U^T_{jo}, \tag{2.24}$$

which represents an eigenvalue problem of the matrix h_{mn} defined as

$$h_{mn} = t_{mn} + \sum_{kl} V_{mknl} \rho^{\mathrm{HF}}_{lk} = \varepsilon_m \delta_{mn}, \tag{2.25}$$

where we rename indices $p \to m, j \to n$. Eq. (2.25) is called the Hartree-Fock equation. By diagonalization of h_{mn} we obtain the self-consistent basis $|i'\rangle$ which is connected to the HO basis $|i\rangle$ by the unitary transformation U_{ij}, see Eq. (2.7a).

The Hamiltonian (2.3) can be rewritten into the form in which the creation and annihilation operators are normal ordered. For this procedure we use the Wick's theorem [45] and obtain the new expression for the Hamiltonian

$$\widehat{H} = E_{\mathrm{HF}} + \widehat{H}^{(1)} + \widehat{H}^{(2)}. \tag{2.26}$$

The HF energy E_{HF} and the operators $\widehat{H}^{(1)}$ and $\widehat{H}^{(2)}$ are expressed as follows:

$$E_{\mathrm{HF}} = \sum_{ij} t_{ij} \rho^{\mathrm{HF}}_{ji} + \frac{1}{2} \sum_{ijkl} V_{ijkl} \rho^{\mathrm{HF}}_{ki} \rho^{\mathrm{HF}}_{jl}, \tag{2.27a}$$

$$\widehat{H}^{(1)} = \sum_{ij} \left(t_{ij} + \sum_{kl} V_{kilj} \rho^{\mathrm{HF}}_{lk} \right) \ :\alpha^\dagger_i \alpha_j:, \tag{2.27b}$$

$$\widehat{H}^{(2)} = \frac{1}{4} \sum_{ijkl} V_{ijkl} \ :\alpha^\dagger_i \alpha^\dagger_j \alpha_l \alpha_k:, \tag{2.27c}$$

where $:\alpha_i^\dagger\alpha_j:$ and $:\alpha_i^\dagger\alpha_j^\dagger\alpha_l\alpha_k:$ denote the normal ordering of operators $\alpha_i^\dagger\alpha_j$ and $\alpha_i^\dagger\alpha_j^\dagger\alpha_l\alpha_k$. Normal ordered operators satisfy

$$\begin{aligned}\langle \mathrm{HF}|:\alpha_i^\dagger\alpha_j:|\mathrm{HF}\rangle &= \langle \mathrm{HF}|:\alpha_i'^\dagger\alpha_j':|\mathrm{HF}\rangle = 0,\\ \langle \mathrm{HF}|:\alpha_i^\dagger\alpha_j^\dagger\alpha_l\alpha_k:|\mathrm{HF}\rangle &= \langle \mathrm{HF}|:\alpha_i'^\dagger\alpha_j'^\dagger\alpha_l'\alpha_k':|\mathrm{HF}\rangle = 0.\end{aligned} \tag{2.28}$$

Therefore, the value of $\langle \mathrm{HF}|\widehat{H}|\mathrm{HF}\rangle$ is equal to the energy of the ground state E_{HF}. The terms $\widehat{H}^{(1)}$ and $\widehat{H}^{(2)}$ in Eq. (2.27b) and (2.27c) represent one- and two-body parts of the Hamiltonian (2.26). The ground-state energy E_{HF} and the operators $\widehat{H}^{(1)}$ and $\widehat{H}^{(2)}$ are expressed in the self-consistent basis as

$$E_{\mathrm{HF}} = \sum_{i-\mathrm{occ.}}\varepsilon_i - \frac{1}{2}\sum_{i,k-\mathrm{occ.}}\overline{V}_{kiki}, \tag{2.29a}$$

$$\widehat{H}^{(1)} = \sum_i \varepsilon_i :\alpha_i'^\dagger\alpha_i':, \tag{2.29b}$$

$$\widehat{H}^{(2)} = \frac{1}{4}\sum_{ijkl}\overline{V}_{ijkl}:\alpha_i'^\dagger\alpha_j'^\dagger\alpha_l'\alpha_k':. \tag{2.29c}$$

Here, $\overline{V}_{ijkl}$ denote the matrix elements V_{ijkl} transformed into the self-consistent basis

$$\overline{V}_{ijkl} = \langle i'j'|\widehat{V}(\vec{r}_1,\vec{r}_2)|k'l'-l'k'\rangle = \sum_{opqr} V_{opqr}U^*_{oi}U^*_{pj}U^T_{lr}U^T_{kq}. \tag{2.30}$$

2.2 Hartree-Fock method in the proton-neutron-Λ formalism with three-body interactions

In this section, we generalize the Hartree-Fock method for a single-Λ hypernucleus consisting of Z protons, N neutrons and one Λ particle. We use the notation ${}^A_\Lambda X = (Z,N,1)$, where $A = Z+N+1$. The subsystem ${}^{(A-1)}X = (Z,N,0)$, where $(A-1) = Z+N$, is called the nuclear core.

Proton-neutron-Λ (p-n-Λ) formalism adopts creation and annihilation operators $a^\dagger, a$ for protons, $b^\dagger, b$ for neutrons and $c^\dagger, c$ for the Λ particle. The kinetic energy and potential operators in the hypernuclear Hamiltonian (2.1) are defined as the following sums of operators:

$$\widehat{T}^N = \widehat{T}^{\mathrm{p}} + \widehat{T}^{\mathrm{n}}, \tag{2.31a}$$

$$\widehat{V}^{NN} = \widehat{V}^{\mathrm{pp}} + \widehat{V}^{\mathrm{pn}} + \widehat{V}^{\mathrm{nn}}, \tag{2.31b}$$

$$\widehat{V}^{N\Lambda} = \widehat{V}^{\mathrm{p}\Lambda} + \widehat{V}^{\mathrm{n}\Lambda}, \tag{2.31c}$$

$$\widehat{V}^{NNN} = \widehat{V}^{\mathrm{ppp}} + \widehat{V}^{\mathrm{nnn}} + \widehat{V}^{\mathrm{ppn}} + \widehat{V}^{\mathrm{pnn}}, \tag{2.31d}$$

$$\widehat{V}^{NN\Lambda} = \widehat{V}^{\mathrm{pp}\Lambda} + \widehat{V}^{\mathrm{nn}\Lambda} + \widehat{V}^{\mathrm{pn}\Lambda}. \tag{2.31e}$$

The Hamiltonian (2.1) of the hypernuclear system in the second quantization in the p-n-Λ formalism reads

$$\begin{aligned}\widehat{H} = &\sum_{ij} t^{\mathrm{p}}_{ij} a^\dagger_i a_j + \sum_{ij} t^{\mathrm{n}}_{ij} b^\dagger_i b_j + \sum_{ij} t^{\Lambda}_{ij} c^\dagger_i c_j \\ &+ \frac{1}{4}\sum_{ijkl} V^{\mathrm{pp}}_{ijkl} a^\dagger_i a^\dagger_j a_l a_k + \frac{1}{4}\sum_{ijkl} V^{\mathrm{nn}}_{ijkl} b^\dagger_i b^\dagger_j b_l b_k + \sum_{ijkl} V^{\mathrm{pn}}_{ijkl} a^\dagger_i a^\dagger_j a_l a_k \\ &+ \sum_{ijkl} V^{\mathrm{p}\Lambda}_{ijkl} a^\dagger_i c^\dagger_j c_l a_k + \sum_{ijkl} V^{\mathrm{n}\Lambda}_{ijkl} b^\dagger_i c^\dagger_j c_l b_k \\ &+ \frac{1}{36}\sum_{ijklmn} V^{\mathrm{ppp}}_{ijklmn} a^\dagger_i a^\dagger_j a^\dagger_k a_n a_m a_l + \frac{1}{36}\sum_{ijklmn} V^{\mathrm{nnn}}_{ijklmn} b^\dagger_i b^\dagger_j b^\dagger_k b_n b_m b_l \\ &+ \frac{1}{4}\sum_{ijklmn} V^{\mathrm{ppn}}_{ijklmn} a^\dagger_i a^\dagger_j b^\dagger_k b_n a_m a_l + \frac{1}{4}\sum_{ijklmn} V^{\mathrm{pnn}}_{ijklmn} a^\dagger_i b^\dagger_j b^\dagger_k b_n b_m a_l \\ &+ \frac{1}{4}\sum_{ijklmn} V^{\mathrm{pp}\Lambda}_{ijklmn} a^\dagger_i a^\dagger_j c^\dagger_k c_n a_m a_l + \frac{1}{4}\sum_{ijklmn} V^{\mathrm{nn}\Lambda}_{ijklmn} b^\dagger_i b^\dagger_j c^\dagger_k c_n b_m b_l \\ &+ \sum_{ijklmn} V^{\mathrm{pn}\Lambda}_{ijklmn} a^\dagger_i b^\dagger_j c^\dagger_k c_n b_m a_l. \end{aligned} \tag{2.32}$$

The matrix elements of the kinetic energy operator, as well as the interaction terms in the HO basis (2.32) are given in Appendix A.

The ground state is described by the ket

$$|\Psi_0\rangle = |\Psi_0\rangle_{\mathrm{p}} \otimes |\Psi_0\rangle_{\mathrm{n}} \otimes |\Psi_0\rangle_{\Lambda}, \tag{2.33}$$

where $|\Psi_0\rangle_{\mathrm{p}}$ and $|\Psi_0\rangle_{\mathrm{n}}$ are Slater determinants of protons and neutrons and $|\Psi_0\rangle_{\Lambda}$ is the lowest single-particle state of the Λ hyperon. Respective states in Eq. (2.33) are constructed as follows

$$|\Psi_0\rangle_{\mathrm{p}} = \prod_{i=1}^{Z} a^\dagger_i |0\rangle, \tag{2.34a}$$

$$|\Psi_0\rangle_{\mathrm{n}} = \prod_{i=1}^{N} b^\dagger_i |0\rangle, \tag{2.34b}$$

$$|\Psi_0\rangle_{\Lambda} = c^\dagger_1 |0\rangle, \tag{2.34c}$$

where the indices i and j run over Z and N lowest occupied states in the proton and the neutron potential wells, respectively. Unitary transformations for each type

of creation (annihilation) operators are

$$a_i'^{\dagger} = \sum_{ij} A_{ij} a_j^{\dagger}; \quad a_i' = \sum_{ij} a_j A_{ij}^{*}, \tag{2.35a}$$

$$b_i'^{\dagger} = \sum_{ij} B_{ij} b_j^{\dagger}; \quad b_i' = \sum_{ij} b_j B_{ij}^{*}, \tag{2.35b}$$

$$c_i'^{\dagger} = \sum_{ij} C_{ij} c_j^{\dagger}; \quad c_i' = \sum_{ij} c_j C_{ij}^{*}. \tag{2.35c}$$

Ground-state in the self-consistent basis is the ket

$$|\mathrm{HF}\rangle = |\mathrm{HF}\rangle_{\mathrm{p}} \otimes |\mathrm{HF}\rangle_{\mathrm{n}} \otimes |\mathrm{HF}\rangle_{\Lambda}, \tag{2.36}$$

where

$$|\mathrm{HF}\rangle_{\mathrm{p}} = \prod_{i=1}^{Z} a_i'^{\dagger} |0\rangle, \tag{2.37a}$$

$$|\mathrm{HF}\rangle_{\mathrm{n}} = \prod_{i=1}^{N} b_i'^{\dagger} |0\rangle, \tag{2.37b}$$

$$|\mathrm{HF}\rangle_{\Lambda} = c_1'^{\dagger} |0\rangle, \tag{2.37c}$$

$$\tag{2.37d}$$

and the density matrices read

$$\rho_{ij}^{\mathrm{p}} = {}_{\mathrm{p}}\langle \mathrm{HF}| a_i^{\dagger} a_j |\mathrm{HF}\rangle_{\mathrm{p}}, \tag{2.38a}$$

$$\rho_{ij}^{\mathrm{n}} = {}_{\mathrm{n}}\langle \mathrm{HF}| b_i^{\dagger} b_j |\mathrm{HF}\rangle_{\mathrm{n}}, \tag{2.38b}$$

$$\rho_{ij}^{\Lambda} = {}_{\Lambda}\langle \mathrm{HF}| c_i^{\dagger} c_j |\mathrm{HF}\rangle_{\Lambda}. \tag{2.38c}$$

We construct the energy functional $\langle \mathrm{HF}|\widehat{H}|\mathrm{HF}\rangle$ as follows:

$$\begin{aligned}
\langle \mathrm{HF}|\widehat{H}|\mathrm{HF}\rangle = &\sum_{ij} t^{\mathrm{p}}_{ij}\langle \mathrm{HF}|a^\dagger_i a_j|\mathrm{HF}\rangle + \sum_{ij} t^{\mathrm{n}}_{ij}\langle \mathrm{HF}|b^\dagger_i b_j|\mathrm{HF}\rangle + \sum_{ij} t^{\Lambda}_{ij}\langle \mathrm{HF}|c^\dagger_i c_j|\mathrm{HF}\rangle \\
&+ \frac{1}{4}\sum_{ijkl} V^{\mathrm{pp}}_{ijkl}\langle \mathrm{HF}|a^\dagger_i a^\dagger_j a_l a_k|\mathrm{HF}\rangle + \frac{1}{4}\sum_{ijkl} V^{\mathrm{nn}}_{ijkl}\langle \mathrm{HF}|b^\dagger_i b^\dagger_j b_l b_k|\mathrm{HF}\rangle \\
&+ \sum_{ijkl} V^{\mathrm{pn}}_{ijkl}\langle \mathrm{HF}|a^\dagger_i b^\dagger_j b_l a_k|\mathrm{HF}\rangle + \sum_{ijkl} V^{\mathrm{p}\Lambda}_{ijkl}\langle \mathrm{HF}|a^\dagger_i c^\dagger_j c_l a_k|\mathrm{HF}\rangle \\
&+ \sum_{ijkl} V^{\mathrm{n}\Lambda}_{ijkl}\langle \mathrm{HF}|b^\dagger_i c^\dagger_j c_l b_k|\mathrm{HF}\rangle \\
&+ \frac{1}{36}\sum_{ijklmn} V^{\mathrm{ppp}}_{ijklmn}\langle \mathrm{HF}|a^\dagger_i a^\dagger_j a^\dagger_k a_n a_m a_l|\mathrm{HF}\rangle \\
&+ \frac{1}{36}\sum_{ijklmn} V^{\mathrm{nnn}}_{ijklmn}\langle \mathrm{HF}|b^\dagger_i b^\dagger_j b^\dagger_k b_n b_m b_l|\mathrm{HF}\rangle \\
&+ \frac{1}{4}\sum_{ijklmn} V^{\mathrm{ppn}}_{ijklmn}\langle \mathrm{HF}|a^\dagger_i a^\dagger_j b^\dagger_k b_n a_m a_l|\mathrm{HF}\rangle \\
&+ \frac{1}{4}\sum_{ijklmn} V^{\mathrm{pnn}}_{ijklmn}\langle \mathrm{HF}|a^\dagger_i b^\dagger_j b^\dagger_k b_n b_m a_l|\mathrm{HF}\rangle \\
&+ \frac{1}{4}\sum_{ijklmn} V^{\mathrm{pp}\Lambda}_{ijklmn}\langle \mathrm{HF}|a^\dagger_i a^\dagger_j c^\dagger_k c_n a_m a_l|\mathrm{HF}\rangle \\
&+ \frac{1}{4}\sum_{ijklmn} V^{\mathrm{nn}\Lambda}_{ijklmn}\langle \mathrm{HF}|b^\dagger_i b^\dagger_j c^\dagger_k c_n b_m b_l|\mathrm{HF}\rangle \\
&+ \sum_{ijklmn} V^{\mathrm{pn}\Lambda}_{ijklmn}\langle \mathrm{HF}|a^\dagger_i b^\dagger_j c^\dagger_k c_n b_m a_l|\mathrm{HF}\rangle. \qquad (2.39)
\end{aligned}$$

We calculate the following identities for the three-body terms:

$$\begin{aligned}
\langle \mathrm{HF}|a^\dagger_i a^\dagger_j a^\dagger_k a_n a_m a_l|\mathrm{HF}\rangle = &(A^TA^*)_{li}(A^TA^*)_{mj}(A^TA^*)_{nk} - (A^TA^*)_{nk}(A^TA^*)_{lj}(A^TA^*)_{mi} \\
&- (A^TA^*)_{mk}(A^TA^*)_{nj}(A^TA^*)_{li} - (A^TA^*)_{mk}(A^TA^*)_{lj}(A^TA^*)_{li} \\
&+ (A^TA^*)_{nj}(A^TA^*)_{lk}(A^TA^*)_{mi} - (A^TA^*)_{lk}(A^TA^*)_{mj}(A^TA^*)_{ni}, \qquad (2.40)
\end{aligned}$$

$$\begin{aligned}
\langle \mathrm{HF}|b^\dagger_i b^\dagger_j b^\dagger_k b_n b_m b_l|\mathrm{HF}\rangle = &(B^TB^*)_{li}(B^TB^*)_{mj}(B^TB^*)_{nk} - (B^TB^*)_{nk}(B^TB^*)_{lj}(B^TB^*)_{mi} \\
&- (B^TB^*)_{mk}(B^TB^*)_{nj}(B^TB^*)_{li} - (B^TB^*)_{mk}(B^TB^*)_{lj}(B^TB^*)_{li} \\
&+ (B^TB^*)_{nj}(B^TB^*)_{lk}(B^TB^*)_{mi} - (B^TB^*)_{lk}(B^TB^*)_{mj}(B^TB^*)_{ni}. \qquad (2.41)
\end{aligned}$$

The terms which contain multiple types of particles satisfy

$$\langle \mathrm{HF}|a_i^\dagger a_j^\dagger b_k^\dagger b_n a_m a_l|\mathrm{HF}\rangle = {}_{\mathrm{p}}\langle \mathrm{HF}|a_i^\dagger a_j^\dagger a_m a_l|\mathrm{HF}\rangle_{\mathrm{p}}\,{}_{\mathrm{n}}\langle \mathrm{HF}|b_k^\dagger b_n|\mathrm{HF}\rangle_{\mathrm{n}}, \tag{2.42a}$$

$$\langle \mathrm{HF}|a_i^\dagger b_j^\dagger c_k^\dagger c_n b_m a_l|\mathrm{HF}\rangle = {}_{\mathrm{p}}\langle \mathrm{HF}|a_i^\dagger a_l|\mathrm{HF}\rangle_{\mathrm{p}}\,{}_{\mathrm{n}}\langle \mathrm{HF}|b_j^\dagger b_m|\mathrm{HF}\rangle_{\mathrm{n}}\,{}_{\Lambda}\langle \mathrm{HF}|c_k^\dagger c_n|\mathrm{HF}\rangle_{\Lambda}. \tag{2.42b}$$

By minimizing the energy functional (2.39) with respect to the unitary transformations A, B, and C we obtain three Hartree-Fock equations – one for each type of particle. The HF equation for protons:

$$\begin{aligned} &t_{ij}^{\mathrm{p}} + \sum_{kl} V_{ikjl}^{\mathrm{pp}}\rho_{lk}^{\mathrm{p}} + \sum_{kl} V_{ikjl}^{\mathrm{pn}}\rho_{lk}^{\mathrm{n}} + \sum_{kl} V_{ikjl}^{\mathrm{p}\Lambda}\rho_{lk}^{\Lambda} + \frac{1}{2}\sum_{klmn} V_{ikljmn}^{\mathrm{ppp}}\rho_{mk}^{\mathrm{p}}\rho_{nl}^{\mathrm{p}} \\ &+ \frac{1}{2}\sum_{klmn} V_{ikljmn}^{\mathrm{pnn}}\rho_{mk}^{\mathrm{n}}\rho_{nl}^{\mathrm{n}} + \sum_{klmn} V_{ikljmn}^{\mathrm{ppn}}\rho_{mk}^{\mathrm{p}}\rho_{nl}^{\mathrm{n}} + \sum_{klmn} V_{ikljmn}^{\mathrm{pp}\Lambda}\rho_{mk}^{\mathrm{p}}\rho_{nl}^{\Lambda} \\ &+ \sum_{klmn} V_{ijklmn}^{\mathrm{pn}\Lambda}\rho_{mk}^{\mathrm{n}}\rho_{nl}^{\Lambda} = \varepsilon_i^{\mathrm{p}}\delta_{ij}. \end{aligned} \tag{2.43}$$

The HF equation for neutrons:

$$\begin{aligned} &t_{ij}^{\mathrm{n}} + \sum_{kl} V_{ikjl}^{\mathrm{nn}}\rho_{lk}^{\mathrm{n}} + \sum_{kl} V_{kilj}^{\mathrm{pn}}\rho_{lk}^{\mathrm{p}} + \sum_{kl} V_{ikjl}^{\mathrm{n}\Lambda}\rho_{lk}^{\Lambda} + \frac{1}{2}\sum_{klmn} V_{ikljmn}^{\mathrm{nnn}}\rho_{mk}^{\mathrm{n}}\rho_{nl}^{\mathrm{n}} \\ &+ \frac{1}{2}\sum_{klmn} V_{klimnj}^{\mathrm{ppn}}\rho_{mk}^{\mathrm{p}}\rho_{nl}^{\mathrm{p}} + \sum_{klmn} V_{klimnj}^{\mathrm{pnn}}\rho_{mk}^{\mathrm{p}}\rho_{nl}^{\mathrm{n}} + \sum_{klmn} V_{ikljmn}^{\mathrm{nn}\Lambda}\rho_{mk}^{\mathrm{n}}\rho_{nl}^{\Lambda} \\ &+ \sum_{klmn} V_{klimnj}^{\mathrm{pn}\Lambda}\rho_{mk}^{\mathrm{p}}\rho_{nl}^{\Lambda} = \varepsilon_i^{\mathrm{n}}\delta_{ij}. \end{aligned} \tag{2.44}$$

The HF equation for the Λ hyperon:

$$\begin{aligned} &t_{ij}^{\Lambda} + \sum_{kl} V_{kilj}^{\mathrm{p}\Lambda}\rho_{lk}^{\mathrm{p}} + \sum_{kl} V_{kilj}^{\mathrm{n}\Lambda}\rho_{lk}^{\mathrm{n}} + \frac{1}{2}\sum_{klmn} V_{klimnj}^{\mathrm{pp}\Lambda}\rho_{mk}^{\mathrm{p}}\rho_{nl}^{\mathrm{p}} + \frac{1}{2}\sum_{klmn} V_{klimnj}^{\mathrm{nn}\Lambda}\rho_{mk}^{\mathrm{n}}\rho_{nl}^{\mathrm{n}} \\ &+ \sum_{klmn} V_{klimnj}^{\mathrm{pn}\Lambda}\rho_{mk}^{\mathrm{p}}\rho_{nl}^{\mathrm{n}} = \varepsilon_i^{\Lambda}\delta_{ij}. \end{aligned} \tag{2.45}$$

After solving the Hartree-Fock Eqs. (2.43), (2.44), and (2.45) we obtain three self-consistent bases associated with the operators $(a'^\dagger, a')$, $(b'^\dagger, b')$, $(c'^\dagger, c')$. Using the Wick's theorem [45] on the hypernuclear Hamiltonian (2.32) gives us the Hamiltonian

$$\widehat{H} = E_{\mathrm{HF}} + \widehat{H}^{(1)} + \widehat{H}^{(2)} + \widehat{H}^{(3)}, \tag{2.46}$$

where E_{HF} is the ground-state (HF) energy, $\widehat{H}^{(1)}$ is a one-body operator, $\widehat{H}^{(2)}$ is a two-body operator, and $\widehat{H}^{(3)}$ is a three-body operator. Here, the HF energy is

expressed as

$$
\begin{aligned}
E_{\mathrm{HF}} = & \sum_{i-\mathrm{occ.}} \varepsilon_i^{\mathrm{p}} + \sum_{i-\mathrm{occ.}} \varepsilon_i^{\mathrm{n}} + \sum_{i-\mathrm{occ.}} \varepsilon_i^{\Lambda} - \frac{1}{2} \sum_{ij-\mathrm{occ.}} \overline{V}_{ijij}^{\mathrm{pp}} - \frac{1}{2} \sum_{ij-\mathrm{occ.}} \overline{V}_{ijij}^{\mathrm{nn}} - \sum_{ij-\mathrm{occ.}} \overline{V}_{ijij}^{\mathrm{pn}} \\
& - \sum_{ij-\mathrm{occ.}} \overline{V}_{ijij}^{\mathrm{p}\Lambda} - \sum_{ij-\mathrm{occ.}} \overline{V}_{ijij}^{\mathrm{n}\Lambda} - \frac{1}{3} \sum_{ijk-\mathrm{occ.}} \overline{V}_{ijkijk}^{\mathrm{ppp}} - \frac{1}{3} \sum_{ijk-\mathrm{occ.}} \overline{V}_{ijkijk}^{\mathrm{nnn}} - \sum_{ijk-\mathrm{occ.}} \overline{V}_{ijkijk}^{\mathrm{ppn}} \\
& - \sum_{ijk-\mathrm{occ.}} \overline{V}_{ijkijk}^{\mathrm{pnn}} - \sum_{ijk-\mathrm{occ.}} \overline{V}_{ijkijk}^{\mathrm{pp}\Lambda} - \sum_{ijk-\mathrm{occ.}} \overline{V}_{ijkijk}^{\mathrm{nn}\Lambda} - 2 \sum_{ijk-\mathrm{occ.}} \overline{V}_{ijkijk}^{\mathrm{pn}\Lambda}. \qquad (2.47)
\end{aligned}
$$

The one-body operator $\widehat{H}^{(1)}$ reads

$$
\widehat{H}^{(1)} = \sum_i \varepsilon_i^{\mathrm{p}} :a_i'^{\dagger} a_i': + \sum_i \varepsilon_i^{\mathrm{n}} :b_i'^{\dagger} b_i': + \sum_i \varepsilon_i^{\Lambda} :c_i'^{\dagger} c_i':, \qquad (2.48)
$$

the two-body operator $\widehat{H}^{(2)}$ is

$$
\begin{aligned}
\widehat{H}^{(2)} = & \frac{1}{4} \sum_{ijkl} \overline{V}_{ijkl}^{\mathrm{pp}} :a_i'^{\dagger} a_j'^{\dagger} a_l' a_k': + \frac{1}{4} \sum_{ijkl} \overline{V}_{ijkl}^{\mathrm{nn}} :b_i'^{\dagger} b_j'^{\dagger} b_l' b_k': + \sum_{ijkl} \overline{V}_{ijkl}^{\mathrm{pn}} :a_i'^{\dagger} b_j'^{\dagger} b_l' a_k': \\
& + \sum_{ijkl} \overline{V}_{ijkl}^{\mathrm{p}\Lambda} :a_i'^{\dagger} c_j'^{\dagger} c_l' a_k': + \sum_{ijkl} \overline{V}_{ijkl}^{\mathrm{n}\Lambda} :b_i'^{\dagger} c_j'^{\dagger} c_l' b_k': + \frac{1}{4} \sum_{ijkl} \sum_{m-\mathrm{occ.}} \overline{V}_{ijmklm}^{\mathrm{ppp}} :a_i'^{\dagger} a_j'^{\dagger} a_l' a_k': \\
& + \frac{1}{4} \sum_{ijkl} \sum_{m-\mathrm{occ.}} \overline{V}_{ijmklm}^{\mathrm{nnn}} :b_i'^{\dagger} b_j'^{\dagger} b_l' b_k': + \frac{1}{4} \sum_{ijkl} \sum_{m-\mathrm{occ.}} \overline{V}_{ijmklm}^{\mathrm{ppn}} :a_i'^{\dagger} a_j'^{\dagger} a_l' a_k': \\
& + \sum_{ijkl} \sum_{m-\mathrm{occ.}} \overline{V}_{mijmkl}^{\mathrm{ppn}} :a_i'^{\dagger} b_j'^{\dagger} b_l' a_k': + \frac{1}{4} \sum_{ijkl} \sum_{m-\mathrm{occ.}} \overline{V}_{mijmkl}^{\mathrm{pnn}} :b_i'^{\dagger} b_j'^{\dagger} b_l' b_k': \\
& + \sum_{ijkl} \sum_{m-\mathrm{occ.}} \overline{V}_{ijmklm}^{\mathrm{pnn}} :a_i'^{\dagger} b_j'^{\dagger} b_l' a_k': + \frac{1}{4} \sum_{ijkl} \sum_{m-\mathrm{occ.}} \overline{V}_{ijmklm}^{\mathrm{pp}\Lambda} :a_i'^{\dagger} a_j'^{\dagger} a_l' a_k': \\
& + \sum_{ijkl} \sum_{m-\mathrm{occ.}} \overline{V}_{mijmkl}^{\mathrm{pp}\Lambda} :a_i'^{\dagger} c_j'^{\dagger} c_l' a_k': + \frac{1}{4} \sum_{ijkl} \sum_{m-\mathrm{occ.}} \overline{V}_{ijmklm}^{\mathrm{nn}\Lambda} :b_i'^{\dagger} b_j'^{\dagger} b_l' b_k': \\
& + \sum_{ijkl} \sum_{m-\mathrm{occ.}} \overline{V}_{mijmkl}^{\mathrm{nn}\Lambda} :b_i'^{\dagger} c_j'^{\dagger} c_l' b_k': + \sum_{ijkl} \sum_{m-\mathrm{occ.}} \overline{V}_{ijmklm}^{\mathrm{pn}\Lambda} :a_i'^{\dagger} b_j'^{\dagger} b_l' a_k': \\
& + \sum_{ijkl} \sum_{m-\mathrm{occ.}} \overline{V}_{imjkml}^{\mathrm{pn}\Lambda} :a_i'^{\dagger} c_j'^{\dagger} c_l' a_k': + \sum_{ijkl} \sum_{m-\mathrm{occ.}} \overline{V}_{mijmkl}^{\mathrm{pn}\Lambda} :b_i'^{\dagger} c_j'^{\dagger} c_l' b_k':, \qquad (2.49)
\end{aligned}
$$

and the three-body part $\widehat{H}^{(3)}$ is

$$
\begin{aligned}
\widehat{H}^{(3)} = & \frac{1}{36} \sum_{ijklmn} \overline{V}_{ijklmn}^{\mathrm{ppp}} :a_i'^{\dagger} a_j'^{\dagger} a_k'^{\dagger} a_n' a_m' a_l': + \frac{1}{36} \sum_{ijklmn} \overline{V}_{ijklmn}^{\mathrm{nnn}} :b_i'^{\dagger} b_j'^{\dagger} b_k'^{\dagger} b_n' b_m' b_l': \\
& + \frac{1}{4} \sum_{ijklmn} \overline{V}_{ijklmn}^{\mathrm{ppn}} :a_i'^{\dagger} a_j'^{\dagger} b_k'^{\dagger} b_n' a_m' a_l': + \frac{1}{4} \sum_{ijklmn} \overline{V}_{ijklmn}^{\mathrm{pnn}} :a_i'^{\dagger} b_j'^{\dagger} b_k'^{\dagger} b_n' b_m' a_l': \\
& + \frac{1}{4} \sum_{ijklmn} \overline{V}_{ijklmn}^{\mathrm{pp}\Lambda} :a_i'^{\dagger} a_j'^{\dagger} c_k'^{\dagger} c_n' a_m' a_l': + \frac{1}{36} \sum_{ijklmn} \overline{V}_{ijklmn}^{\mathrm{nn}\Lambda} :b_i'^{\dagger} b_j'^{\dagger} c_k'^{\dagger} c_n' b_m' b_l': \\
& + \sum_{ijklmn} \overline{V}_{ijklmn}^{\mathrm{pn}\Lambda} :a_i'^{\dagger} b_j'^{\dagger} c_k'^{\dagger} c_n' b_m' a_l': . \qquad (2.50)
\end{aligned}
$$

The interaction matrix elements in Eqs. (2.47), (2.49), (2.50) are represented in the self-consistent basis. I.e. they are transformed from the interaction elements in the HO basis by relations similar to Eq. (2.30).

Chapter 3

Coupling of Λ with nuclear core

In this chapter, we introduce the coupling of the Λ particle with the nuclear core within the framework of the Tamm-Dancoff Approximation (TDA). The extension to the hypernuclear systems is denoted as the nucleon-Λ Tamm-Dancoff Approximation (NΛ TDA).

The NΛ TDA method describes the spectra of hypernuclei consisting of one Λ particle bound in an even-odd nuclear core. The hypernuclear states are constructed by annihilating one nucleon from an even-even nuclear system and coupling of the even-odd core with the Λ particle.

3.1 Tamm-Dancoff approximation

Eq. (2.36) describes the hypernuclear ground state within the mean-field approximation. The general excitations of this state can be represented as the sum of one-, two-, and many- particle-hole excitations of the HF ground state. Here we restrict ourselves to one-particle-hole excitations.

The starting point of the TDA, applied only on the nuclear core, is the self-consistent basis obtained in the HF calculations. The operator which creates the particle-hole excitation is defined as:

$$Q_\mu^\dagger = \sum_{ph} \left(C_{ph}^{\mu,\mathrm{p}} a_p'^\dagger a_{\overline{h}}' + C_{ph}^{\mu,\mathrm{n}} b_p'^\dagger b_{\overline{h}}' \right), \tag{3.1}$$

where $a_p'^\dagger, a_{\overline{h}}'$ ($b_p'^\dagger, b_{\overline{h}}'$) denote the proton (neutron) creation and annihilation operators with respect to the particle (p) and hole (h) states in the self-consistent basis. The operator $Q_\mu^\dagger$ is the phonon operator. We use the following phase convention

$$|\overline{h}\rangle = |\alpha_h \; j_h \; \overline{m}_h\rangle = (-1)^{j_h+m_h} |\alpha_h \; j_h \; -m_h\rangle. \tag{3.2}$$

The need for this convention arises from the fact that the hole single-particle states $|\overline{h}\rangle$ transform under rotations as a jm spherical tensor [46]. The coefficients $C_{ph}^{\mu,\mathrm{p}}$ ($C_{ph}^{\mu,\mathrm{n}}$) in Eq. (3.1) represent the linear combinations of the proton (neutron) particle-hole (ph) excitations.

The derivation of the TDA method starts from the hypernuclear Hamiltonian in Eq. (2.46) with all matrix elements expressed in the self-consistent basis. The HF densities in the self-consistent basis are expressed as:

$$\overline{\rho}^{\mathrm{p}}_{nm} = \langle \mathrm{HF}|a'^{\dagger}_m a'_n|\mathrm{HF}\rangle = \sum_{m-\mathrm{occ.}} \delta_{nm}, \tag{3.3a}$$

$$\overline{\rho}^{\mathrm{n}}_{nm} = \langle \mathrm{HF}|b'^{\dagger}_m b'_n|\mathrm{HF}\rangle = \sum_{m-\mathrm{occ.}} \delta_{nm}, \tag{3.3b}$$

$$\overline{\rho}^{\Lambda}_{nm} = \langle \mathrm{HF}|c'^{\dagger}_m c'_n|\mathrm{HF}\rangle = \sum_{m-\mathrm{occ.}} \delta_{nm}. \tag{3.3c}$$

We introduce the normal ordered interaction elements in the two-body part of the Hamiltonian (2.49)

$$\overline{V}^{\mathrm{pp,gen}}_{ijkl} = \overline{V}^{\mathrm{pp}}_{ijkl} + \sum_{mn}\left[\overline{V}^{\mathrm{ppp}}_{ijmkln}\overline{\rho}^{\mathrm{p}}_{nm} + \overline{V}^{\mathrm{ppn}}_{ijmkln}\overline{\rho}^{\mathrm{n}}_{nm} + \overline{V}^{\mathrm{pp}\Lambda}_{ijmkln}\overline{\rho}^{\Lambda}_{nm}\right], \tag{3.4a}$$

$$\overline{V}^{\mathrm{nn,gen}}_{ijkl} = \overline{V}^{\mathrm{nn}}_{ijkl} + \sum_{mn}\left[\overline{V}^{\mathrm{nnn}}_{ijmkln}\overline{\rho}^{\mathrm{n}}_{nm} + \overline{V}^{\mathrm{nnp}}_{ijmkln}\overline{\rho}^{\mathrm{p}}_{nm} + \overline{V}^{\mathrm{nn}\Lambda}_{ijmkln}\overline{\rho}^{\Lambda}_{nm}\right], \tag{3.4b}$$

$$\overline{V}^{\mathrm{pn,gen}}_{ijkl} = \overline{V}^{\mathrm{pn}}_{ijkl} + \sum_{mn}\left[\overline{V}^{\mathrm{ppn}}_{mijnkl}\overline{\rho}^{\mathrm{p}}_{nm} + \overline{V}^{\mathrm{pnn}}_{ijmkln}\overline{\rho}^{\mathrm{n}}_{nm} + \overline{V}^{\mathrm{pn}\Lambda}_{ijmkln}\overline{\rho}^{\Lambda}_{nm}\right], \tag{3.4c}$$

$$\overline{V}^{\mathrm{p}\Lambda,\mathrm{gen}}_{ijkl} = \overline{V}^{\mathrm{p}\Lambda}_{ijkl} + \sum_{mn}\left[\overline{V}^{\mathrm{pp}\Lambda}_{mijnkl}\overline{\rho}^{\mathrm{p}}_{nm} + \overline{V}^{\mathrm{pn}\Lambda}_{imjknl}\overline{\rho}^{\mathrm{n}}_{nm}\right], \tag{3.4d}$$

$$\overline{V}^{\mathrm{n}\Lambda,\mathrm{gen}}_{ijkl} = \overline{V}^{\mathrm{n}\Lambda}_{ijkl} + \sum_{mn}\left[\overline{V}^{\mathrm{nn}\Lambda}_{mijnkl}\overline{\rho}^{\mathrm{n}}_{nm} + \overline{V}^{\mathrm{pn}\Lambda}_{mijnkl}\overline{\rho}^{\mathrm{p}}_{nm}\right]. \tag{3.4e}$$

The matrix elements $\overline{V}^{\mathrm{pp,gen}}_{ijkl}, \overline{V}^{\mathrm{nn,gen}}_{ijkl}, \overline{V}^{\mathrm{pn,gen}}_{ijkl}, \overline{V}^{\mathrm{p}\Lambda,\mathrm{gen}}_{ijkl}, \overline{V}^{\mathrm{n}\Lambda,\mathrm{gen}}_{ijkl}$ represent the two-body interaction corrected by the presence of the three-body force in the A-body nuclear system. The two-body part $\widehat{H}^{(2)}$ in Eq. (2.49) of the Hamiltonian $\widehat{H}$ is expressed with the help of normal-ordered two-body matrix elements as

$$\begin{aligned}\widehat{H}^{(2)} = &\frac{1}{4}\sum_{ijkl}\overline{V}^{\mathrm{pp,gen}}_{ijkl} : a'^{\dagger}_i a'^{\dagger}_j a'_l a'_k : + \frac{1}{4}\sum_{ijkl}\overline{V}^{\mathrm{nn,gen}}_{ijkl} : b'^{\dagger}_i b'^{\dagger}_j b'_l b'_k : \\ &+ \sum_{ijkl}\overline{V}^{\mathrm{pn,gen}}_{ijkl} : a'^{\dagger}_i b'^{\dagger}_j b'_l a'_k : + \sum_{ijkl}\overline{V}^{\mathrm{p}\Lambda,\mathrm{gen}}_{ijkl} : a'^{\dagger}_i c'^{\dagger}_j c'_k a'_l : \\ &+ \sum_{ijkl}\overline{V}^{\mathrm{n}\Lambda,\mathrm{gen}}_{ijkl} : b'^{\dagger}_i c'^{\dagger}_j c'_l b'_k :\end{aligned} \tag{3.5}$$

The TDA method is based on the following equation of motion

$$\langle \mathrm{HF}|Q_{\nu'}[\widehat{H}, Q^{\dagger}_{\nu}]|\mathrm{HF}\rangle = (E_{\nu} - E_{\mathrm{HF}})\delta_{\nu\nu'}. \tag{3.6}$$

By inserting Eq. (2.46), the commutator $[\widehat{H}, Q_\nu^\dagger]$ in Eq. (3.6) can be evaluated term by term. The HF energy E_{HF} is a number and thus

$$[E_{\mathrm{HF}}, Q_\nu^\dagger]|\mathrm{HF}\rangle = 0. \tag{3.7}$$

Furthermore, we note that

$$[\widehat{H}^{(1)}, Q_\nu^\dagger]|\mathrm{HF}\rangle = \sum_{ph}(\varepsilon_p^{\mathrm{p}} - \varepsilon_h^{\mathrm{p}})C_{ph}^{\nu,\mathrm{p}} a_p'^\dagger a_{\overline{h}}'|\mathrm{HF}\rangle + \sum_{ph}(\varepsilon_p^{\mathrm{n}} - \varepsilon_h^{\mathrm{n}})C_{ph}^{\nu,\mathrm{n}} b_p'^\dagger b_{\overline{h}}'|\mathrm{HF}\rangle. \tag{3.8}$$

The two-body part of the Hamiltonian commutes with the phonon operator:

$$\begin{aligned}[\widehat{H}^{(2)}, Q_\nu^\dagger]|\mathrm{HF}\rangle = \Bigg(&- \sum_{ph}\sum_{p_1h_1} \overline{V}^{\mathrm{pp,gen}}_{p_1\overline{h}ph_1} C_{ph}^{\nu,\mathrm{p}} a_{p_1}'^\dagger a_{h_1}' - \sum_{ph}\sum_{p_1h_1} \overline{V}^{\mathrm{nn,gen}}_{p_1\overline{h}ph_1} C_{ph}^{\nu,\mathrm{n}} b_{p_1}'^\dagger b_{h_1}' \\ &+ \sum_{ph}\sum_{p_1h_1} \overline{V}^{\mathrm{pn,gen}}_{\overline{h}p_1ph_1} C_{ph}^{\nu,\mathrm{p}} b_{p_1}'^\dagger b_{h_1}' + \sum_{ph}\sum_{p_1h_1} \overline{V}^{\mathrm{pn,gen}}_{\overline{h}p_1ph_1} C_{ph}^{\nu,\mathrm{n}} a_{p_1}'^\dagger a_{h_1}' \Bigg)|\mathrm{HF}\rangle. \end{aligned} \tag{3.9}$$

The term $[\widehat{H}^{(3)}, Q_\nu^\dagger]|\mathrm{HF}\rangle$ does not contribute to the TDA equation,

$$\langle \mathrm{HF}|Q_{\nu'}[\widehat{H}^{(3)}, Q_\nu^\dagger]|\mathrm{HF}\rangle = 0. \tag{3.10}$$

Inserting (3.8) and (3.9) into the TDA Eq. (3.6) leads to the eigenvalue problem

$$\sum_{ph} \begin{pmatrix} (\varepsilon_p^{\mathrm{p}} - \varepsilon_h^{\mathrm{p}})\delta_{pp'}\delta_{hh'} + \overline{V}^{\mathrm{pn,gen}}_{p'\overline{h}\overline{h}'p} & \overline{V}^{\mathrm{pn,gen}}_{p'\overline{h}\overline{h}'p} \\ \overline{V}^{\mathrm{pn,gen}}_{\overline{h}p'p\overline{h}'} & (\varepsilon_p^{\mathrm{p}} - \varepsilon_h^{\mathrm{p}})\delta_{pp'}\delta_{hh'} + \overline{V}^{\mathrm{pn,gen}}_{p'\overline{h}\overline{h}'p} \end{pmatrix} \begin{pmatrix} C_{ph}^{\nu,\mathrm{p}} \\ \\ C_{ph}^{\nu,\mathrm{n}} \end{pmatrix} = (E_\nu - E_{\mathrm{HF}}) \begin{pmatrix} C_{p'h'}^{\nu,\mathrm{p}} \\ \\ C_{p'h'}^{\nu,\mathrm{n}} \end{pmatrix}. \tag{3.11}$$

It should be noted that the operators $\overline{V}^{\mathrm{p}\Lambda,\mathrm{gen}}_{ijkl}$ and $\overline{V}^{\mathrm{n}\Lambda,\mathrm{gen}}_{ijkl}$ get subtracted in the commutators. Therefore, the ΛN interaction does not contribute to the TDA Eq. (3.11).

3.2 NΛ TDA method

The principle of the NΛ TDA method is the annihilation of one nucleon from the even-even nuclear system and coupling of this even-odd (or odd-even) core with the Λ hyperon. In analogy to Eq. (3.1), the NΛ TDA phonon operators are defined as

$$R^\dagger_{\mu,\mathrm{p}\Lambda} = \sum_{ph} r_{ph}^{\mu,\mathrm{p}\Lambda} c_p'^\dagger a_{\overline{h}}', \tag{3.12a}$$

$$R^\dagger_{\mu,\mathrm{n}\Lambda} = \sum_{ph} r_{ph}^{\mu,\mathrm{n}\Lambda} c_p'^\dagger b_{\overline{h}}'. \tag{3.12b}$$

The equations of motion of the NΛ TDA method are formulated in analogy to the TDA:

$$\langle \mathrm{HF}|R_{\nu',\mathrm{p}\Lambda}[\widehat{H}, R^\dagger_{\nu,\mathrm{p}\Lambda}]|\mathrm{HF}\rangle = (E^{\mathrm{p}\Lambda}_\nu - E_{\mathrm{HF}})\delta_{\nu\nu'}, \tag{3.13a}$$

$$\langle \mathrm{HF}|R_{\nu',\mathrm{n}\Lambda}[\widehat{H}, R^\dagger_{\nu,\mathrm{n}\Lambda}]|\mathrm{HF}\rangle = (E^{\mathrm{n}\Lambda}_\nu - E_{\mathrm{HF}})\delta_{\nu\nu'}. \tag{3.13b}$$

By inserting the Hamiltonian $\widehat{H}$ as a sum of terms $E_{\mathrm{HF}}, \widehat{H}^{(1)}, \widehat{H}^{(2)}$, and $\widehat{H}^{(3)}$, we evaluate the commutation relations $[\widehat{H}, R^\dagger_{\nu,\mathrm{p}\Lambda}]$ and $[\widehat{H}, R^\dagger_{\nu,\mathrm{n}\Lambda}]$ term by term

$$[E_{\mathrm{HF}}, R^\dagger_{\nu,\mathrm{p}\Lambda}]|\mathrm{HF}\rangle = 0, \tag{3.14a}$$

$$[E_{\mathrm{HF}}, R^\dagger_{\nu,\mathrm{n}\Lambda}]|\mathrm{HF}\rangle = 0, \tag{3.14b}$$

$$[\widehat{H}^{(1)}, R^\dagger_{\nu,\mathrm{p}\Lambda}]|\mathrm{HF}\rangle = \sum_{ph}(\varepsilon^\Lambda_p - \varepsilon^{\mathrm{p}}_h) r^{\nu,\mathrm{p}\Lambda}_{ph} c'^\dagger_p a'_{\overline{h}}|\mathrm{HF}\rangle, \tag{3.15a}$$

$$[\widehat{H}^{(1)}, R^\dagger_{\nu,\mathrm{n}\Lambda}]|\mathrm{HF}\rangle = \sum_{ph}(\varepsilon^\Lambda_p - \varepsilon^{\mathrm{n}}_h) r^{\nu,\mathrm{n}\Lambda}_{ph} c'^\dagger_p b'_{\overline{h}}|\mathrm{HF}\rangle, \tag{3.15b}$$

$$[\widehat{H}^{(2)}, R^\dagger_{\nu,\mathrm{p}\Lambda}]|\mathrm{HF}\rangle = -\sum_{p_1h_1}\sum_{ph} \overline{V}^{\mathrm{p}\Lambda,\mathrm{gen}}_{\overline{h}p_1h_1p} c'^\dagger_{p_1} a'_{h_1}|\mathrm{HF}\rangle, \tag{3.16a}$$

$$[\widehat{H}^{(2)}, R^\dagger_{\nu,\mathrm{n}\Lambda}]|\mathrm{HF}\rangle = -\sum_{p_1h_1}\sum_{ph} \overline{V}^{\mathrm{n}\Lambda,\mathrm{gen}}_{\overline{h}p_1h_1p} c'^\dagger_{p_1} b'_{h_1}|\mathrm{HF}\rangle. \tag{3.16b}$$

Again, it applies that

$$\langle \mathrm{HF}|R_{\nu',\mathrm{p}\Lambda}[\widehat{H}^{(3)}, R^\dagger_{\nu,\mathrm{p}\Lambda}]|\mathrm{HF}\rangle = \langle \mathrm{HF}|R_{\nu',\mathrm{n}\Lambda}[\widehat{H}^{(3)}, R^\dagger_{\nu,\mathrm{n}\Lambda}]|\mathrm{HF}\rangle = 0. \tag{3.17a}$$

The substitution of Eqs. (3.15a) and (3.16a) into Eq. (3.13a) (and Eqs. (3.15b) and (3.16b) into Eq. (3.13b)) leads to

$$\langle \mathrm{HF}|R_{\nu',\mathrm{p}\Lambda}[\widehat{H}, R^\dagger_{\nu,\mathrm{p}\Lambda}]|\mathrm{HF}\rangle = \sum_{ph}\sum_{p'h'} r^{\nu',\mathrm{p}\Lambda}_{p'h'} r^{\nu,\mathrm{p}\Lambda}_{ph} \left((\varepsilon^\Lambda_p - \varepsilon^{\mathrm{p}}_h)\delta_{pp'}\delta_{hh'} - \overline{V}^{\mathrm{p}\Lambda,\mathrm{gen}}_{\overline{h}p'\overline{h}'p}\right), \tag{3.18a}$$

$$\langle \mathrm{HF}|R_{\nu',\mathrm{n}\Lambda}[\widehat{H}, R^\dagger_{\nu,\mathrm{n}\Lambda}]|\mathrm{HF}\rangle = \sum_{ph}\sum_{p'h'} r^{\nu',\mathrm{n}\Lambda}_{p'h'} r^{\nu,\mathrm{n}\Lambda}_{ph} \left((\varepsilon^\Lambda_p - \varepsilon^{\mathrm{n}}_h)\delta_{pp'}\delta_{hh'} - \overline{V}^{\mathrm{n}\Lambda,\mathrm{gen}}_{\overline{h}p'\overline{h}'p}\right). \tag{3.18b}$$

Eqs. (3.18a) and (3.18b) correspond to the following eigenvalue problems:

$$\sum_{ph}\left((\varepsilon^\Lambda_p - \varepsilon^{\mathrm{p}}_h)\delta_{pp'}\delta_{hh'} - \overline{V}^{\mathrm{p}\Lambda,\mathrm{gen}}_{\overline{h}p'\overline{h}'p}\right) r^{\nu,\mathrm{p}\Lambda}_{ph} = (E^{\mathrm{p}\Lambda}_\nu - E_{\mathrm{HF}}) r^{\nu,\mathrm{p}\Lambda}_{p'h'}, \tag{3.19a}$$

$$\sum_{ph}\left((\varepsilon^\Lambda_p - \varepsilon^{\mathrm{n}}_h)\delta_{pp'}\delta_{hh'} - \overline{V}^{\mathrm{n}\Lambda,\mathrm{gen}}_{\overline{h}p'\overline{h}'p}\right) r^{\nu,\mathrm{n}\Lambda}_{ph} = (E^{\mathrm{n}\Lambda}_\nu - E_{\mathrm{HF}}) r^{\nu,\mathrm{n}\Lambda}_{p'h'}. \tag{3.19b}$$

In practice, $\overline{V}^{\mathrm{p(n)}\Lambda,\mathrm{gen}}_{ijkl} \equiv \overline{V}^{\mathrm{p(n)}\Lambda}_{ijkl}$. We have derived the N$\Lambda$ TDA method with the three-body ΛNN interactions but we have not yet implemented the ΛNN interactions in the numerical calculations.

3.3 General coupling of Λ with multi-particle-hole excitations of nuclear core

In Chapter 2, we derived the Hartree-Fock method in the proton-neutron-Λ formalism which is suitable for description of hypernuclear systems which consist of even-even nuclear core and one Λ hyperon.

In Section 3.1, we discussed the excitations of the even-even nuclear core within the TDA method as a general superposition of one-particle one-hole nucleon excitations of the HF ground state.

In Section 3.2, we generalized the TDA method and derived the NΛ TDA method which can be used for the calculations of hypernuclei with even-odd (or odd-even) nuclear core and one bound Λ hyperon.

The methods which are used for calculations of hypernuclear spectra (HF in the p-n-Λ formalism and NΛ TDA) can be understood as a starting point for two different generalizations of the Equation of Motion Phonon Method (EMPM) [24, 25, 26]. Within the EMPM we split the Hilbert space into the direct sum of the n Hilbert subspaces:

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \ldots \oplus \mathcal{H}_n. \tag{3.20}$$

In the first generalization we consider the coupling of the Λ particle with exictations of the nuclear core and thus construct the Hilbert subspaces in the following way:

$$\mathcal{H}_0 = \left\{ c'^{\dagger}_p |\mathrm{HF}\rangle_{\mathrm{p}} \otimes |\mathrm{HF}\rangle_{\mathrm{n}} \right\}, \tag{3.21a}$$

$$\mathcal{H}_1 = \left\{ Q^{\dagger}_{\mu_1} c'^{\dagger}_p |\mathrm{HF}\rangle_{\mathrm{p}} \otimes |\mathrm{HF}\rangle_{\mathrm{n}} \right\}, \tag{3.21b}$$

$$\mathcal{H}_2 = \left\{ Q^{\dagger}_{\mu_1} Q^{\dagger}_{\mu_2} c'^{\dagger}_p |\mathrm{HF}\rangle_{\mathrm{p}} \otimes |\mathrm{HF}\rangle_{\mathrm{n}} \right\}, \tag{3.21c}$$

$$\vdots$$

$$\mathcal{H}_n = \left\{ Q^{\dagger}_{\mu_1} Q^{\dagger}_{\mu_2} \ldots Q^{\dagger}_{\mu_n} c'^{\dagger}_p |\mathrm{HF}\rangle_{\mathrm{p}} \otimes |\mathrm{HF}\rangle_{\mathrm{n}} \right\}. \tag{3.21d}$$

The diagonalization of the hypernuclear Hamiltonian (2.46) in the Hilbert space defined in such way is suitable for description of hypernuclear systems with even-even core and one Λ particle.

In the second generalization we consider the coupling of the NΛ TDA states with the excitations of the nuclear core. The Hilbert subspaces are then generated as follows:

$$\mathcal{H}_0 = \left\{ R^\dagger_{\nu,\mathrm{p(n)}\Lambda} |\mathrm{HF}\rangle_\mathrm{p} \otimes |\mathrm{HF}\rangle_\mathrm{n} \right\}, \tag{3.22a}$$

$$\mathcal{H}_1 = \left\{ Q^\dagger_{\mu_1} R^\dagger_{\nu,\mathrm{p(n)}\Lambda} |\mathrm{HF}\rangle_\mathrm{p} \otimes |\mathrm{HF}\rangle_\mathrm{n} \right\}, \tag{3.22b}$$

$$\mathcal{H}_2 = \left\{ Q^\dagger_{\mu_1} Q^\dagger_{\mu_2} R^\dagger_{\nu,\mathrm{p(n)}\Lambda} |\mathrm{HF}\rangle_\mathrm{p} \otimes |\mathrm{HF}\rangle_\mathrm{n} \right\}, \tag{3.22c}$$

$$\vdots$$

$$\mathcal{H}_n = \left\{ Q^\dagger_{\mu_1} Q^\dagger_{\mu_2} \dots Q^\dagger_{\mu_n} R^\dagger_{\nu,\mathrm{p(n)}\Lambda} |\mathrm{HF}\rangle_\mathrm{p} \otimes |\mathrm{HF}\rangle_\mathrm{n} \right\}. \tag{3.22d}$$

Similarly, diagonalization of the Hamiltonian (2.32) in this Hilbert space is suitable for description of hypernuclei with even-odd (or odd-even) nuclear cores and one Λ particle.

Chapter 4

Implementation of the three-body force

All interaction elements of the NN, ΛN, and the NNN interactions are represented and stored in the J-scheme formalism (see Appendix B). I.e. we work with the J-coupled two-body elements $V^{J,\mathrm{pp}}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l)}$, $V^{J,\mathrm{pn}}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l)}$, $V^{J,\mathrm{nn}}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l)}$, $V^{J,\mathrm{p}\Lambda}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l)}$, $V^{J,\mathrm{n}\Lambda}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l)}$, and with the JT-coupled three-body elements $V^{J'_{12},J_{12},J,T'_{12},T_{12},T}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l),(n_m l_m j_m),(n_n l_n j_n)}$.

We need to place restrictions on the indices i, j, k, l, m, n which enumerate the $(n_\bullet l_\bullet j_\bullet)$ configurations within the NNN matrix elements to reduce the demands on the computer memory. We can use the antisymmetry and store only the matrix elements for the indices

$$i \geq j \geq k, \tag{4.1a}$$

$$l \geq m \geq n. \tag{4.1b}$$

Furthermore, we use the hermiticity of the NNN potential. We introduce another restriction which can be defined as $(ijk) \geq (lmn)$, e.g.

$$(i \cdot 10^6 + j \cdot 10^3 + k) \geq (l \cdot 10^6 + m \cdot 10^3 + n). \tag{4.2}$$

In Eq. (4.2), we suppose that we work within a single-particle basis with dim $< 10^3$.

The matrix elements of the NNN interaction that are not stored due to the restrictions (4.1a), (4.1b), and (4.2) need to be reconstructed on the fly.

In the code, we implement the HF equations for protons and neutrons as follows:

$$
\begin{aligned}
&t^{\mathrm{p(n)}}_{(n_i l_i j_i),(n_j l_j j_j)}\delta_{l_i l_j}\delta_{j_i j_j}\delta_{m_i m_j} \\
&+\sum_J \sum_{\substack{n_k l_k j_k \\ n_l l_l j_l}} V^{J,\mathrm{pp(nn)}}_{(n_i l_i j_i),(n_k l_k j_k),(n_j l_j j_j),(n_l l_l j_l)} \rho^{\mathrm{p(n)}}_{(n_l l_l j_l),(n_k l_k j_k)} \delta_{l_i l_k}\delta_{j_i j_k}\delta_{m_i m_j}\frac{2J+1}{2j_i+1} \\
&+\sum_J \sum_{\substack{n_k l_k j_k \\ n_l l_l j_l}} V^{J,\mathrm{pn(np)}}_{(n_i l_i j_i),(n_k l_k j_k),(n_j l_j j_j),(n_l l_l j_l)} \rho^{\mathrm{n(p)}}_{(n_l l_l j_l),(n_k l_k j_k)} \delta_{l_i l_k}\delta_{j_j j_k}\delta_{m_i m_j}\frac{2J+1}{2j_i+1} \\
&+\sum_J \sum_{\substack{n_k l_k j_k \\ n_l l_l j_l}} V^{J,\mathrm{p\Lambda(n\Lambda)}}_{(n_i l_i j_i),(n_k l_k j_k),(n_j l_j j_j),(n_l l_l j_l)} \rho^{\Lambda(\Lambda)}_{(n_l l_l j_l),(n_k l_k j_k)} \delta_{l_i l_k}\delta_{j_i j_k}\delta_{m_i m_j}\frac{2J+1}{2j_i+1}\frac{1}{2j_k+1} \\
&+\sum_{n_k l_k j_k m_k}\sum_{n_l l_l j_j m_l}\sum_{n_m l_m j_m m_m}\sum_{n_n l_n j_n m_n} \\
&\Bigg(\frac{1}{2}V^{\mathrm{ppp(nnn)}}_{n_i l_i j_i m_i, n_k l_k j_k m_k, n_l l_l j_l m_l, n_j l_j j_j m_j, n_m l_m j_m m_m, n_n l_n j_n m_n} \rho^{\mathrm{p(n)}}_{n_m l_m j_m m_m, n_k l_k j_k m_k} \rho^{\mathrm{p(n)}}_{n_n l_n j_n m_n, n_l l_l j_l m_l} \\
&+\frac{1}{2}V^{\mathrm{pnn(ppn)}}_{n_i l_i j_i m_i, n_k l_k j_k m_k, n_l l_l j_l m_l, n_j l_j j_j m_j, n_m l_m j_m m_m, n_n l_n j_n m_n} \rho^{\mathrm{n(p)}}_{n_m l_m j_m m_m, n_k l_k j_k m_k} \rho^{\mathrm{n(n)}}_{n_n l_n j_n m_n, n_l l_l j_l m_l} \\
&+V^{\mathrm{ppn(pnn)}}_{n_i l_i j_i m_i, n_k l_k j_k m_k, n_l l_l j_l m_l, n_j l_j j_j m_j, n_m l_m j_m m_m, n_n l_n j_n m_n} \rho^{\mathrm{p(n)}}_{n_m l_m j_m m_m, n_k l_k j_k m_k} \rho^{\mathrm{n(n)}}_{n_n l_n j_n m_n, n_l l_l j_l m_l}\Bigg) \\
&=\varepsilon^{\mathrm{p(n)}}_i \delta_{ij}. \qquad (4.3)
\end{aligned}
$$

The elements of the NNN interactions in Eq. (4.3) are decoupled into the M-scheme from the JT-coupled elements on the fly by using the transformation equations (B.9),(B.10), (B.11), and (B.12). In addition, the following HF equation for the Λ hyperon is implemented,

$$
\begin{aligned}
&t^{\Lambda}_{(n_i l_i j_i),(n_j l_j j_j)}\delta_{l_i l_j}\delta_{j_i j_j}\delta_{m_i m_j} \\
&+\sum_J \sum_{\substack{n_k l_k j_k \\ n_l l_l j_l}} V^{J,\mathrm{p\Lambda}}_{(n_k l_k j_k),(n_i l_i j_i),(n_l l_l j_l),(n_j l_j j_j)} \rho^{\mathrm{p}}_{(n_l l_l j_l),(n_k l_k j_k)} \delta_{l_k l_l}\delta_{j_k j_l}\delta_{m_i m_j}\frac{2J+1}{2j_i+1} \\
&+\sum_J \sum_{\substack{n_k l_k j_k \\ n_l l_l j_l}} V^{J,\mathrm{n\Lambda}}_{(n_k l_k j_k),(n_i l_i j_i),(n_l l_l j_l),(n_j l_j j_j)} \rho^{\mathrm{n}}_{(n_l l_l j_l),(n_k l_k j_k)} \delta_{l_k l_l}\delta_{j_k j_l}\delta_{m_i m_j}\frac{2J+1}{2j_i+1} \\
&=\varepsilon^{\Lambda}_i \delta_{ij}. \qquad (4.4)
\end{aligned}
$$

The HF code can run either in the static or in the dynamic mode. The code in the static mode first solves Eqs. for protons and neutrons (4.3) without the proton-Λ and the neutron-Λ interactions. Afterwards, it solves Eq. (4.4) for the Λ hyperon. In the static mode, the properties of protons and neutrons in the nuclear core are not affected by the presence of the Λ hyperon. We use the static mode for calculations of the nuclear core and as a starting point for the NΛ TDA calculations. The code in the dynamic mode solves Eqs. (4.3) and (4.4) self-consistently for all particle species. We use the dynamic mode for calculations of the Λ single-particle spectra.

The HF method is implemented in the izotropic HO basis. This basis is infinite. In practical calculations, the basis is truncated by the maximal oscillator shell number $N_{\max}$. I.e. the single-particle configuration space is spanned by the following set of the single-particle states $\{|i\rangle : 2n_i + l_i = N_i \leq N_{\max}\}$. The number n_i stands for the principal quantum number and l_i is the orbital angular momentum. The two-body interaction operators are represented by matrix elements of product states $|ij\rangle = |i\rangle|j\rangle$, $|kl\rangle = |k\rangle|l\rangle$ (see equations (A.7a), (A.7b)). The two-particle basis is truncated consistently with the single-particle basis by the number $N_{\max}^{(12)}$. I.e. the two-particle configuration space is spanned by a set of the states $\{|ij\rangle : 2n_i + l_i + 2n_j + l_j = N_i + N_j \leq 2N_{\max} = N_{\max}^{(12)}\}$. Similar logic applies to the three-body operators which are represented by matrix elements of product states $|ijk\rangle = |i\rangle|j\rangle|k\rangle$, $|lmn\rangle = |l\rangle|m\rangle|n\rangle$. The three-particle basis is truncated by the number $N_{\max}^{(123)}$. The configuration space is spanned by a set of states $\{|ijk\rangle : 2n_i + l_i + 2n_j + l_j + 2n_k + l_k = N_i + N_j + N_k \leq 3N_{\max} = N_{\max}^{(123)}\}$. The conditions $N_{\max}^{(12)} = 2N_{\max}$ and $N_{\max}^{(123)} = 3N_{\max}$ lead to computational problems for any configuration space bigger than $N_{\max} = 5$. In such space, it is very complicated to generate, store, and operate with all three-body interaction matrix elements. For this reason we constrain the configuration space by the following condition $N_{\max} = N_{\max}^{(12)} = N_{\max}^{(123)}$.

The width of the potential well of the spherical harmonic oscillator is given by the parameter $\hbar\omega$. In this work, the parameter $\hbar\omega$ is fixed to 16 MeV.

In the NΛ TDA method, the three-body NNN interaction does not enter the equations explicitly as in the case of the HF method in the p-n-Λ formalism. The residual three-body interaction $\widehat{H}^{(3)}$ from Eq. (2.50) does not contribute to NΛ TDA Eqs. (3.19a) and (3.19b), respectively. The NNN force contributes only to the generalized two-body interaction terms presented in Eqs. (3.4a) - (3.4e). These matrix elements are called generalized interaction elements.

Chapter 5

Results

In this work, we study the ground-state properties of nuclei ^{16}O, ^{40}Ca, ^{48}Ca, and the energy spectra of single-Λ hypernuclei ${}^{17}_{\Lambda}$O, ${}^{41}_{\Lambda}$Ca, ${}^{49}_{\Lambda}$Ca, ${}^{16}_{\Lambda}$O, ${}^{40}_{\Lambda}$Ca, ${}^{48}_{\Lambda}$Ca. We calculate the radial density distributions, charge radii, nucleon single-particle spectra, and binding energies of ^{16}O, ^{40}Ca, and ^{48}Ca in the mean-field approximation by solving the Hartree-Fock (HF) Eq. (4.3) in the static mode. The single-Λ hypernuclei ${}^{17}_{\Lambda}$O, ${}^{41}_{\Lambda}$Ca, and ${}^{49}_{\Lambda}$Ca, which consist of one Λ hyperon bound to the even-even nuclear core, are computed by the HF method in the proton-neutron-Λ (p-n-Λ) formalism. We calculate their Λ single-particle spectra by solving the HF Eqs. (4.3) and (4.4) in the dynamic mode. The single-Λ hypernuclei ${}^{16}_{\Lambda}$O, ${}^{40}_{\Lambda}$Ca, and ${}^{48}_{\Lambda}$Ca consist of one Λ particle bound to the even-odd nuclear core. We generate their energy spectra first by solving the HF Eqs. (4.3) and (4.4) in the static mode and then solving the NΛ TDA Eq. (3.19b).

We employ the chiral N^{2}LO$_{\rm sat}$ NN and NNN interaction [32] which represents the force acting among nucleons. The force acting between Λ hyperons and nucleons is described by the NΛ–NΛ channel of the chiral LO YN interaction [39].

The configuration space was truncated by $N_{\rm max}$. The parameter $\hbar\omega$ was set to 16 MeV for all calculations.

5.1 Calculations of ^{16}O, ^{40}Ca, and ^{48}Ca

In this section, we study the nuclei ^{16}O, ^{40}Ca, and ^{48}Ca within the mean-field approximation. First we check the convergence of the radial density distributions, charge radii, and the nucleon single-particle energies with respect to $N_{\rm max}$. Then we study the effect of the NNN force on these observables.

The radial density distribution $\rho(r)$ is defined as follows:

$$\rho(r) = \rho_{\rm p}(r) + \rho_{\rm n}(r), \tag{5.1}$$

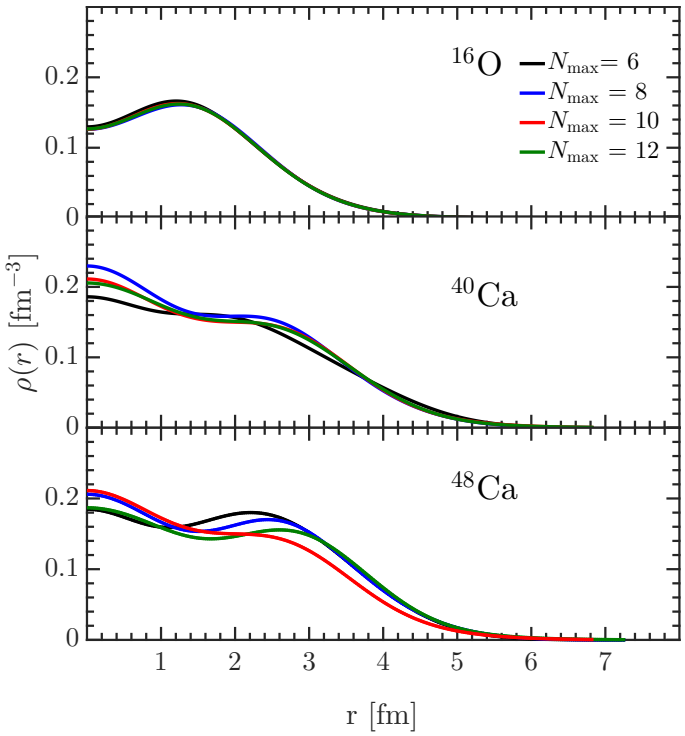

Fig. 5.1: The radial density distributions $\rho(r)$ of the nuclei ^{16}O, ^{40}Ca, and ^{48}Ca calculated with NN + NNN interaction with $N_{\max} = 6, 8, 10, 12$.

where $\rho_{\mathrm{p}}(r)$ and $\rho_{\mathrm{n}}(r)$ are proton and neutron density distributions, respectively. We calculate the proton radial density distribution $\rho_{\mathrm{p}}(r)$ as:

$$\rho_{\mathrm{p}}(r) = \frac{1}{4\pi} \sum_{ij} \sum_{k-\mathrm{occ.}} R_{n_i l_i}(r,b) R_{n_j l_j}(r,b) A_{kj} A_{ki} (2j_k + 1). \tag{5.2}$$

The neutron radial density distribution $\rho_{\mathrm{n}}(r)$ is calculated from the expression:

$$\rho_{\mathrm{n}}(r) = \frac{1}{4\pi} \sum_{ij} \sum_{k-\mathrm{occ.}} R_{n_i l_i}(r,b) R_{n_j l_j}(r,b) B_{kj} B_{ki} (2j_k + 1). \tag{5.3}$$

The functions $R_{n_i l_i}(r,b)$ in Eqs. (5.2) and (5.3) are the radial parts of the HO wave functions and are discussed in detail in Appendix A. The matrices A and B represent the unitary transformations between the HO and the self-consistent bases introduced in Chapter 2.

In Fig. 5.1, the radial density distributions of ^{16}O, ^{40}Ca, and ^{48}Ca for $N_{\max} = 6, 8, 10$, and 12 are plotted. These density distributions were calculated with NN and NNN interactions. In ^{16}O, the density distributions exhibit rapid convergence. In ^{40}Ca, the curves show slower convergence, and in ^{48}Ca, the convergence is not reached, since bigger configuration space is required.

In Figs. 5.2, 5.3, and 5.4, the radial density distributions of ^{16}O, ^{40}Ca, and ^{48}Ca, calculated with and without the NNN force, along with the ones calculated with

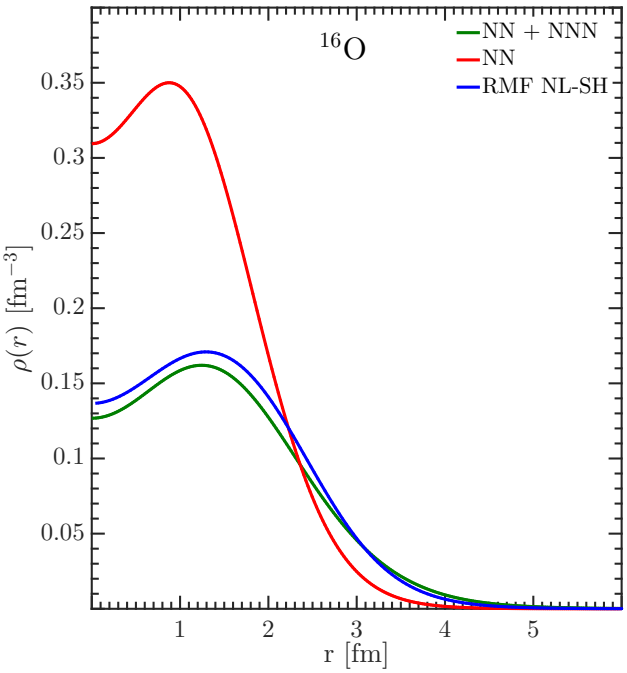

Fig. 5.2: The radial density distribution $\rho(r)$ of ^{16}O calculated only with NN interaction (red line), with NN + NNN interaction (green line), and with the RMF model NL-SH [48] (blue line).

the relativistic mean-field (RMF) NL-SH model [48], are shown. The RMF is a phenomenological model which reproduces empirical density distributions [49]. The density distributions calculated within our model are the converged ones obtained for $N_{\max} = 12$. The NNN force has a repulsive effect which flattens and expands the density distributions. Moreover, the calculations with NNN interaction yield results in better agreement with the empirical density distributions computed by the RMF.

Similar effect of the NNN force is observed for the charge radii. The mean-square charge radius $\langle r^2_{\rm ch}\rangle$ of a nucleus ${}^{A}_{Z}\mathrm{X}_{N}$ is calculated from standard expression [32]:

$$\langle r^2_{\rm ch}\rangle = \left(1 - \frac{1}{A}\right)\langle r^2_{\rm p}\rangle + R^2_{\rm p} + \frac{N}{Z}R^2_{\rm n} + \frac{3\hbar^2}{4m^2_{\rm p}c^2}, \tag{5.4}$$

where $R_{\rm p} = 0.8775(51)$ fm, $R^2_{\rm n} = 0.1149(27)$ fm^2, and $\frac{3\hbar^2}{4m^2_{\rm p}c^2} \sim 0.033$ fm^2. The term $\langle r^2_{\rm p}\rangle$ in Eq. (5.4) is the point proton mean-square radius

$$\langle r^2_{\rm p}\rangle = \frac{\int \mathrm{d}r\; r^4\rho_{\rm p}(r)}{\int \mathrm{d}r\; r^2\rho_{\rm p}(r)}. \tag{5.5}$$

The results are shown in Table 5.1. The quantity $\langle r_{\rm ch}\rangle$ is calculated as $\langle r_{\rm ch}\rangle = \sqrt{\langle r^2_{\rm ch}\rangle}$. Nuclear charge radii calculated only with the NN interaction are unrealistically compressed, whereas charge radii calculated with the NN + NNN interaction are in better agreement with the available experimental data.

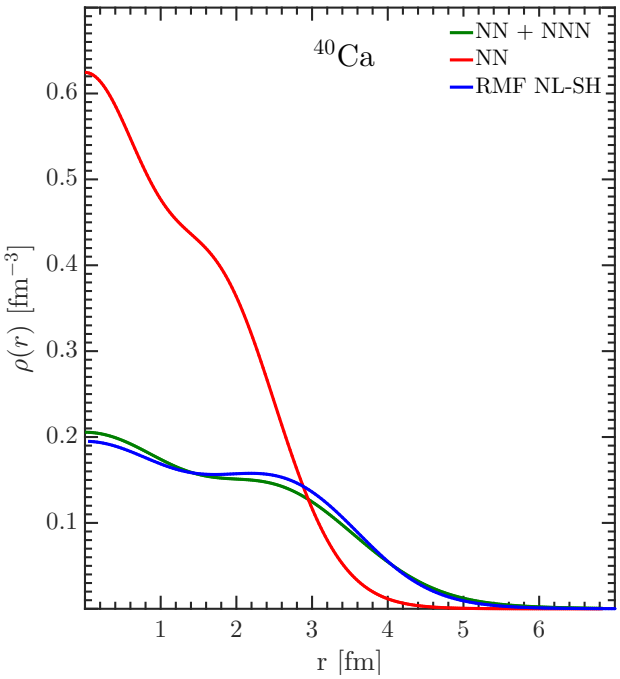

Fig. 5.3: The radial density distribution $\rho(r)$ of ^{40}Ca calculated only with NN interaction (red line), with NN + NNN interaction (green line), and with the RMF model NL-SH [48] (blue line).

In the HF method, the value E_{HF} approximates the ground-state energy of a given nucleus. The calculated binding energy per nucleon is:

$$\frac{BE}{A} = -\frac{E_{\mathrm{HF}}}{A}. \tag{5.6}$$

The binding energies per nucleon of the ^{16}O, ^{40}Ca, and ^{48}Ca are shown in Table 5.2. The repulsive character of the NNN force decreases significantly the binding energy to unrealistic values. Generally, the binding energies calculated by the HF method from the realistic nucleon interactions do not reproduce the experimental data [50].

Table 5.1: The charge radii $\langle r_{\mathrm{ch}} \rangle$ of the ^{16}O, ^{40}Ca, and ^{48}Ca calculated only with NN and with NN + NNN interaction, compared to the experimental data (exp) taken from [51].

	$\langle r_{\mathrm{ch}} \rangle$ [fm]		
^{A}X	NN	NN + NNN	exp
^{16}O	2.24	2.96	2.70
^{40}Ca	2.62	3.68	3.48
^{48}Ca	2.59	3.60	3.47

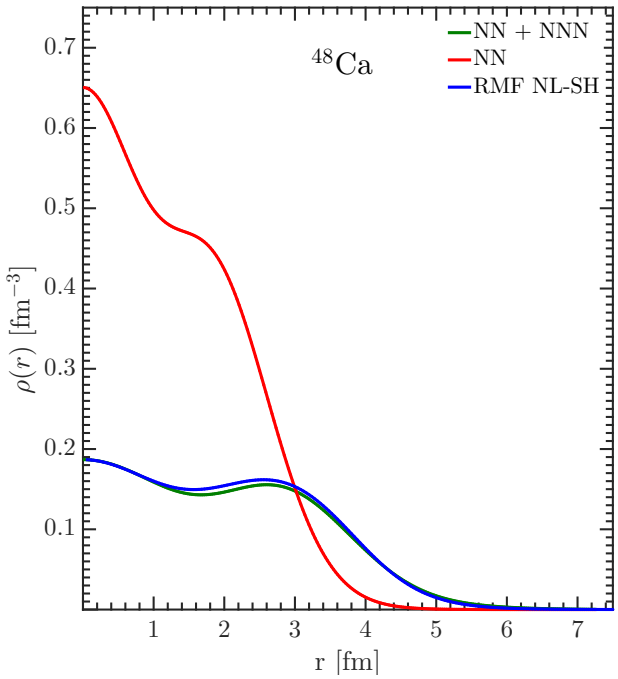

Fig. 5.4: The radial density distribution $\rho(r)$ of ^{48}Ca calculated only with NN interaction (red line), with NN + NNN interaction (green line), and with the RMF model NL-SH [48] (blue line).

In order to obtain realistic binding energies, we need to implement short-range and many-body correlations.

Next, we study the convergence of the neutron single-particle energies $\varepsilon_i^{\mathrm{n}}$. The neutron single-particle energies calculated with the NN + NNN interaction are shown in Fig. 5.5. For all states which lie under the Fermi level, the convergence is reached. However, the convergence is much slower for all unoccupied levels. The convergence properties of the proton single-particle energies $\varepsilon_i^{\mathrm{p}}$ are similar to the neutron ones. Therefore, we do not present them in this thesis.

Table 5.2: Binding energies per nucleon BE/A calculated with NN interaction and with NN + NNN interaction in ^{16}O, ^{40}Ca, and ^{48}Ca compared to the experimental values (exp).

	BE/A [MeV]		
^{A}X	NN	NN + NNN	exp
^{16}O	7.36	2.66	7.98
^{40}Ca	11.65	2.31	8.55
^{48}Ca	12.95	1.93	8.67

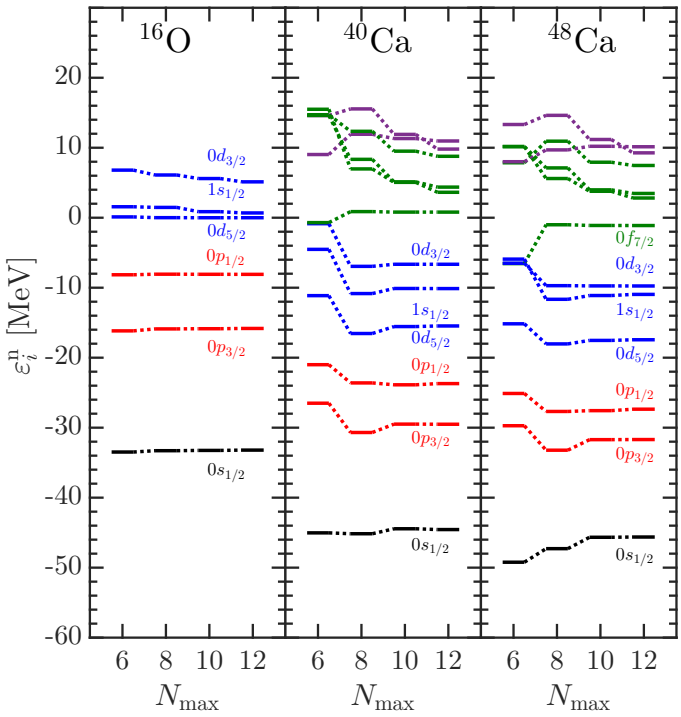

Fig. 5.5: The neutron single-particle energies $\varepsilon_i^{\mathrm{n}}$ of the nuclei ^{16}O, ^{40}Ca, and ^{48}Ca calculated with NN + NNN interaction with $N_{\mathrm{max}} = 6, 8, 10, 12$.

In Figs. 5.6, 5.7, and 5.8, the neutron single-particle energies $\varepsilon_i^{\mathrm{n}}$ of the ^{16}O, ^{40}Ca, and ^{48}Ca calculated only with NN interaction are compared to the ones calculated with the NN + NNN interaction. The parameter N_{max} is fixed to 12. The empirical energies extracted from experimental data are shown as well.

The empirical values of the binding energies are determined from the differences between binding energies of doubly-magic nuclei ^{16}O, ^{40}Ca, and ^{48}Ca and the corresponding neighboring odd-even nuclei. The single-particle energies of the unoccupied levels are calculated by the equations

$$\varepsilon^{\mathrm{n}}(^{16}\mathrm{O}) = BE(^{16}\mathrm{O}) - BE(^{17}\mathrm{O}), \tag{5.7a}$$
$$\varepsilon^{\mathrm{n}}(^{40}\mathrm{Ca}) = BE(^{40}\mathrm{Ca}) - BE(^{41}\mathrm{Ca}), \tag{5.7b}$$
$$\varepsilon^{\mathrm{n}}(^{48}\mathrm{Ca}) = BE(^{48}\mathrm{Ca}) - BE(^{49}\mathrm{Ca}), \tag{5.7c}$$

where $B(^{A}\mathrm{X})$ is the binding energy of the given nucleus. The single-particle energies of the occupied levels are obtained by

$$\varepsilon^{\mathrm{n}}(^{16}\mathrm{O}) = BE(^{15}\mathrm{O}) - BE(^{16}\mathrm{O}), \tag{5.8a}$$
$$\varepsilon^{\mathrm{n}}(^{40}\mathrm{Ca}) = BE(^{39}\mathrm{Ca}) - BE(^{40}\mathrm{Ca}), \tag{5.8b}$$
$$\varepsilon^{\mathrm{n}}(^{48}\mathrm{Ca}) = BE(^{47}\mathrm{Ca}) - BE(^{48}\mathrm{Ca}). \tag{5.8c}$$

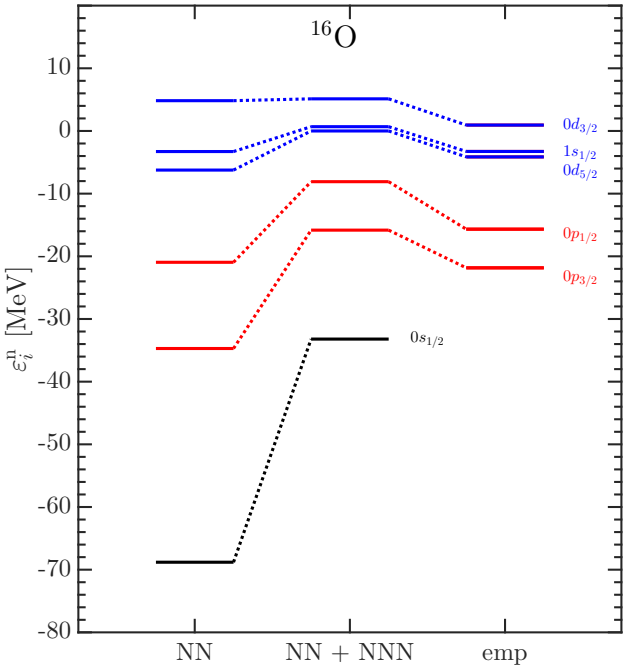

Fig. 5.6: The neutron single-particle energies $\varepsilon_i^{\mathrm{n}}$ of ^{16}O calculated with only NN interaction and with NN + NNN interaction. The empirical data (emp) [52] are shown for comparison.

The repulsive NNN interaction quenches the gaps between the major shells. The calculations with NN + NNN interaction yield results in better agreement with the available empirical data [52].

The proton (neutron) density ρ_{p} (ρ_{n}) influences the Λ single-particle energies through the HF Eq. (4.4). The proton (neutron) single-particle energies $\varepsilon_i^{\mathrm{p}}$ ($\varepsilon_i^{\mathrm{n}}$) have main impact on the hypernuclear spectrum in NΛ TDA Eqs. (3.19a) and (3.19b). Therefore, for calculations of hypernuclear spectra, it is most important to improve the description of density distributions and nucleon single-particle energies in nuclear cores. We observe that NNN interaction within the potential N^{2}LO$_{\mathrm{sat}}$ improves description of the mentioned nuclear observables.

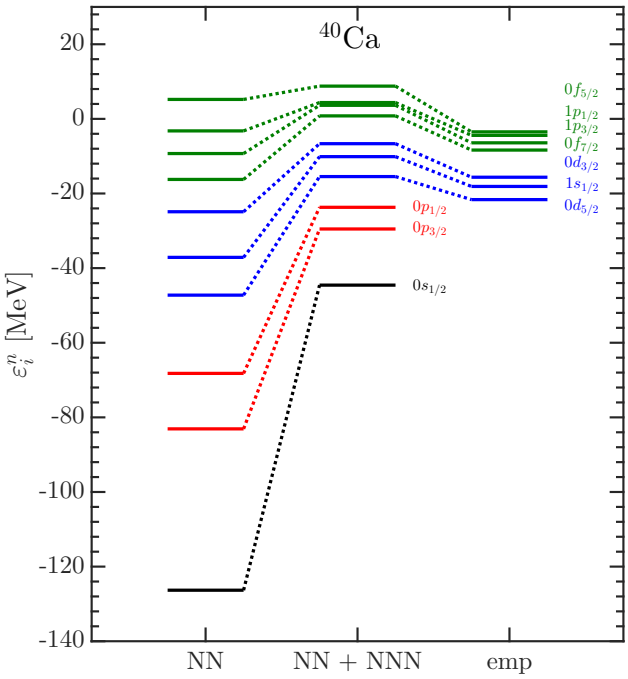

Fig. 5.7: The neutron single-particle energies $\varepsilon_i^{\mathrm{n}}$ of ^{40}Ca calculated with only NN interaction and with NN + NNN interaction. The empirical data (emp) [52] are shown for comparison.

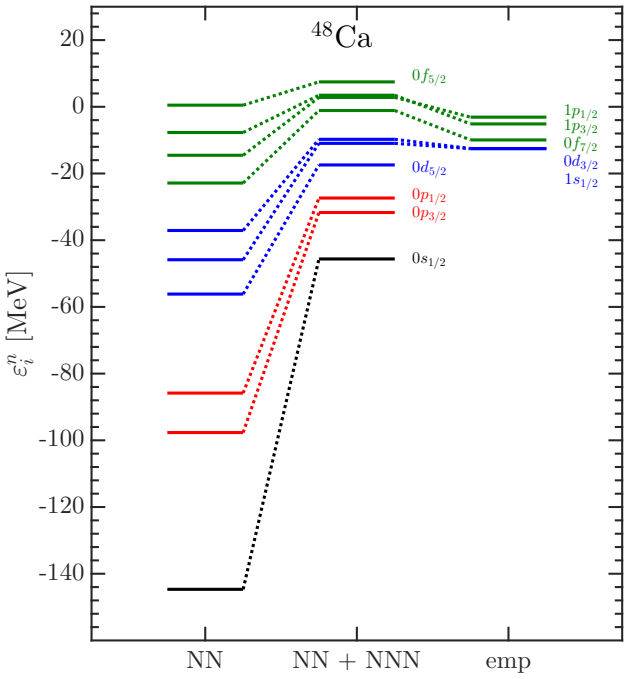

Fig. 5.8: The neutron single-particle energies $\varepsilon_i^{\mathrm{n}}$ of ^{48}Ca calculated with only NN interaction and with NN + NNN interaction. The empirical data (emp) [52] are shown for comparison.

5.2 Calculations of $^{17}_{\Lambda}$O, $^{41}_{\Lambda}$Ca, and $^{49}_{\Lambda}$Ca

The hypernuclei $^{17}_{\Lambda}$O, $^{41}_{\Lambda}$Ca, and $^{49}_{\Lambda}$Ca consist of one Λ particle bound in the even-even nuclear cores ^{16}O, ^{40}Ca, and ^{48}Ca, respectively. We calculate the Λ single-particle energies in these hypernuclei by the HF method in the p-n-Λ formalism solved in the dynamic mode (see Chapter 4).

In Fig. 5.9, the Λ single-particle energies in $^{17}_{\Lambda}$O, $^{41}_{\Lambda}$Ca, and $^{49}_{\Lambda}$Ca, calculated with $N_{\max} = 6, 8, 10$, and 12, are shown. The Λ single-particle states with negative energies represent the bound states of the Λ hyperon and reach quick convergence in all considered hypernuclei. Convergence of the Λ single-particle states with positive energies is slower.

In Figs. 5.10, 5.11, and 5.12 there are Λ single-particle energies ε_i^{Λ} in the hypernuclei $^{17}_{\Lambda}$O, $^{41}_{\Lambda}$Ca, and $^{49}_{\Lambda}$Ca calculated with and without NNN interaction. The gaps among major shells are strongly quenched in calculations with NN + NNN force. The results in $^{17}_{\Lambda}$O and $^{41}_{\Lambda}$Ca are compared to the available empirical data. In the calculations with NN + NNN interaction, the Λ single-particle energies are systematically shifted upwards in energy with respect to empirical data, but the relative energies between the major shells are in better agreement with these data.

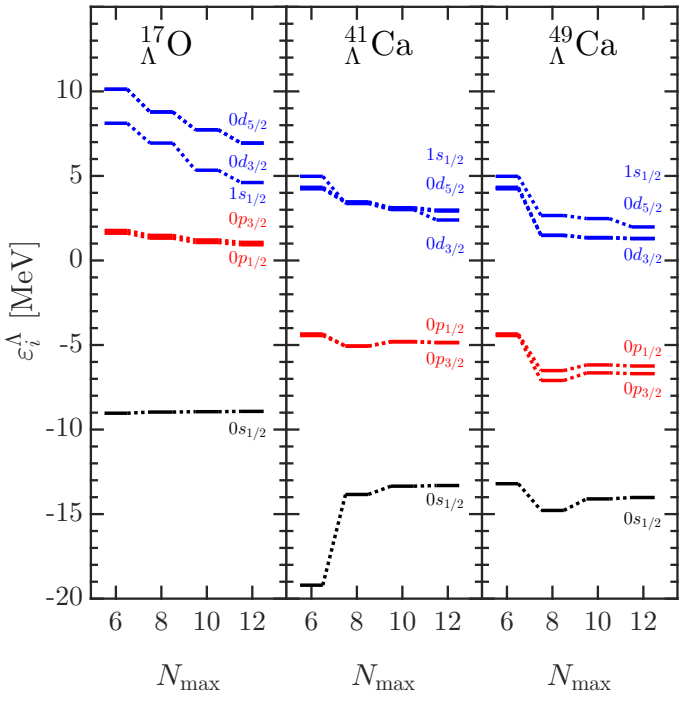

Fig. 5.9: The Λ single-particle energies ε_i^{Λ} in $^{17}_{\Lambda}$O, $^{41}_{\Lambda}$Ca, and $^{49}_{\Lambda}$Ca calculated with NN + NNN interaction with $N_{\max} = 6, 8, 10$, and 12.

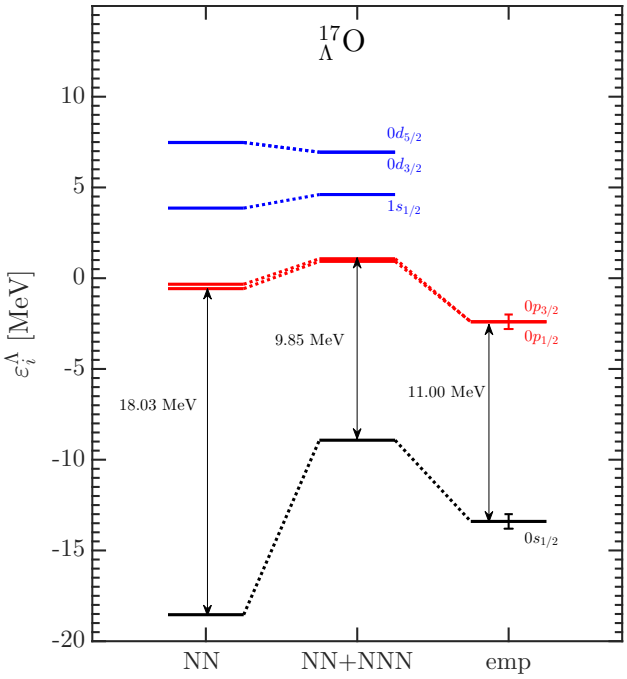

Fig. 5.10: The Λ single-particle energies ε_i^Λ of $^{17}_{\Lambda}$O calculated with only NN interaction and with NN + NNN interaction. The empirical data (emp) interpreted from the experimental measurements [53] are shown for comparison.

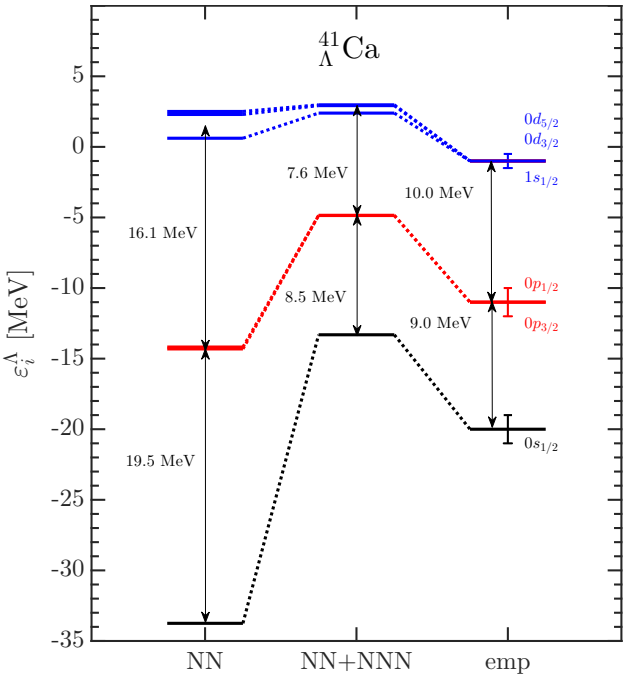

Fig. 5.11: The Λ single-particle energies ε_i^Λ of $^{41}_{\Lambda}$Ca calculated with only NN interaction and with NN + NNN interaction. The empirical data (emp) interpreted from the experimental measurements [54] are shown for comparison.

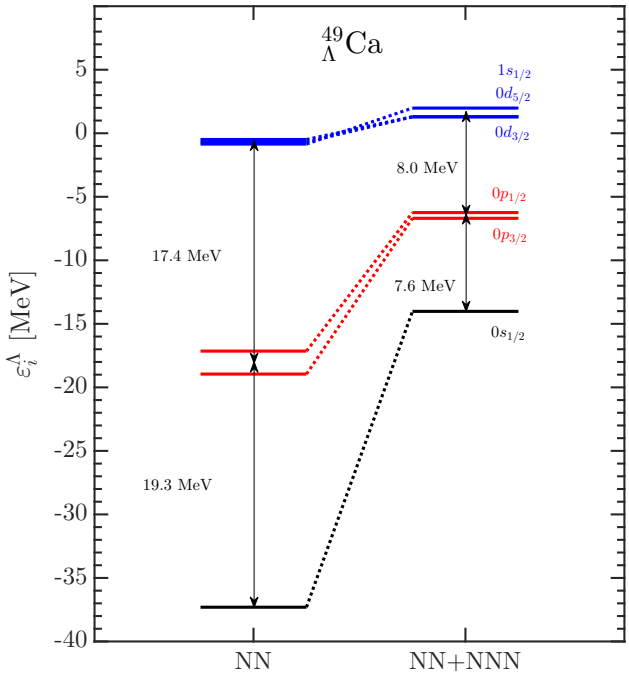

Fig. 5.12: The Λ single-particle energies ε_i^Λ of $^{49}_{\Lambda}$Ca calculated with only NN interaction and with NN + NNN interaction.

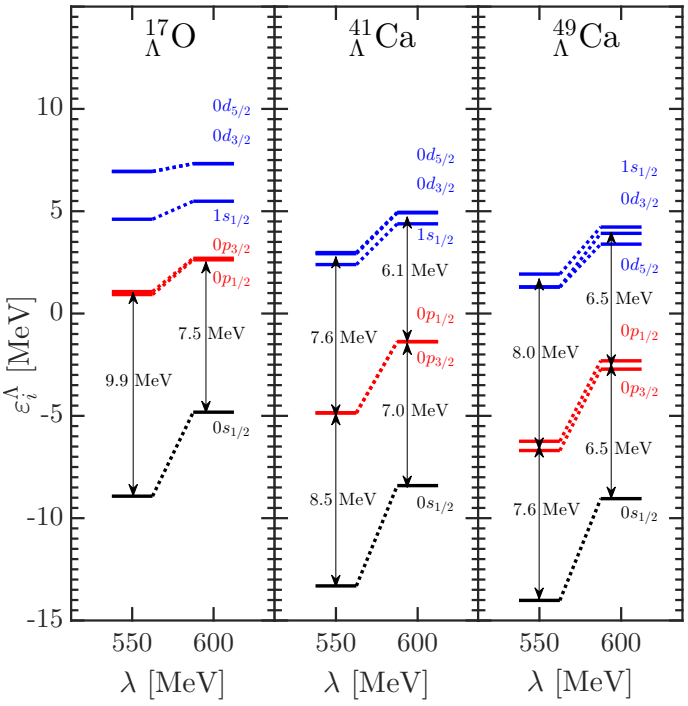

Fig. 5.13: The Λ single-particle energies ε_i^Λ of $^{17}_{\Lambda}$O, $^{41}_{\Lambda}$Ca, and $^{49}_{\Lambda}$Ca calculated with NN + NNN interaction for cutoff parameter of the YN interaction $\lambda = 550$ and 600 MeV.

The YN interaction used in our model is derived only in the leading order and thus it is strongly dependent on the cutoff parameter λ [39]. In Fig. 5.13, the Λ single-particle energies of ${}^{17}_{\Lambda}$O, ${}^{41}_{\Lambda}$Ca, and ${}^{49}_{\Lambda}$Ca calculated with $\lambda = 550$ and 600 MeV in configuration space truncated by $N_{\max} = 12$ are shown. We observe that the Λ single-particle energy spectra of all hypernuclei are shifted upwards by using the YN interaction with higher cutoff λ. The relative energies between the single-particle levels depend on λ as well. However, this dependence is very small in comparison to the influence of the NNN force which we discuss in the text.

5.3 Calculations of ${}^{16}_{\Lambda}$O, ${}^{40}_{\Lambda}$Ca, and ${}^{48}_{\Lambda}$Ca

The NΛ TDA method is used for calculations of hypernuclei with Λ hyperon bound to even-odd nuclear core. Such hypernuclei are typically produced in experiments through (π^+, K^+) reactions [55]. We present calculations of the energy spectra of medium-mass single-Λ hypernuclei ${}^{16}_{\Lambda}$O, ${}^{40}_{\Lambda}$Ca, and ${}^{48}_{\Lambda}$Ca.

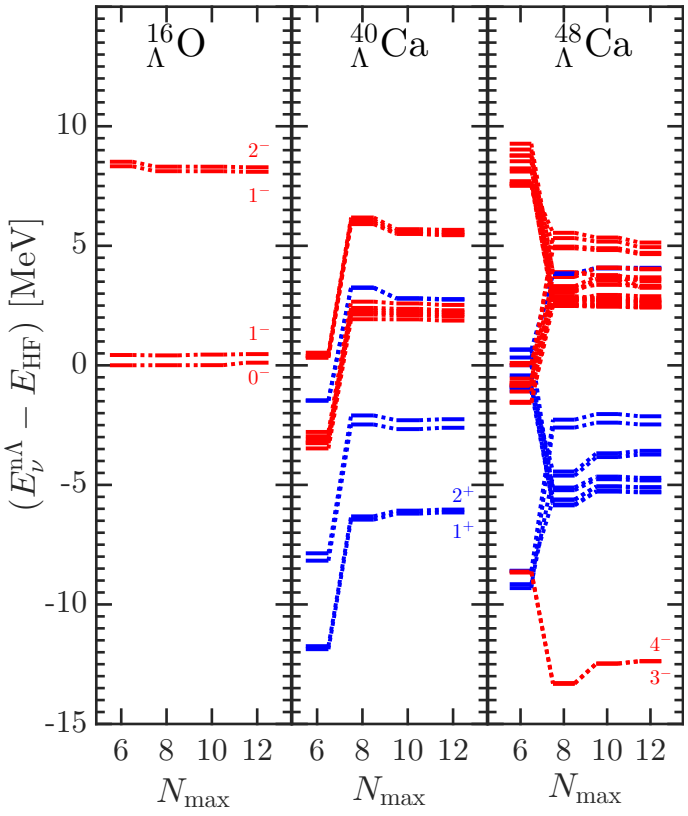

Fig. 5.14: The relative energies $(E_\nu^{n\Lambda} - E_{HF})$ of ${}^{16}_{\Lambda}$O, ${}^{40}_{\Lambda}$Ca, and ${}^{48}_{\Lambda}$Ca calculated with NN + NNN interaction with $N_{\max} = 6, 8, 10$, and 12.

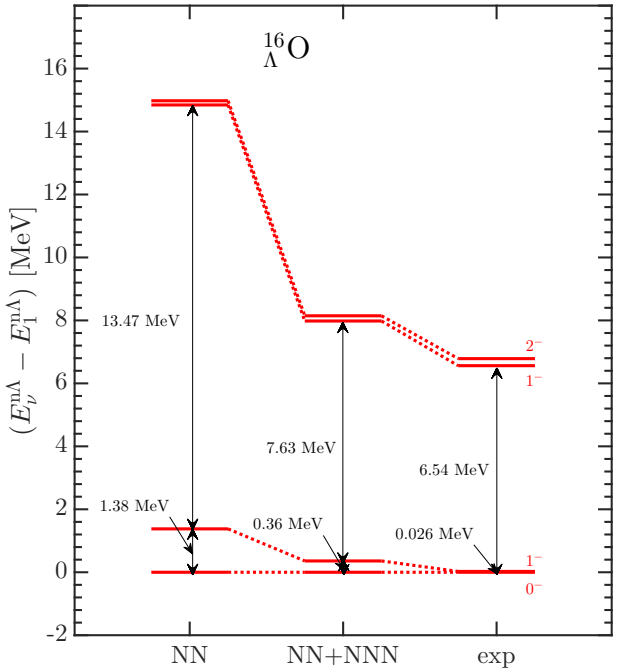

Fig. 5.15: The relative energies $(E_\nu^{n\Lambda} - E_1^{n\Lambda})$ of $^{16}_{\Lambda}$O calculated with respect to the lowest energy level of $^{16}_{\Lambda}$O with only NN interaction and with NN + NNN interaction. The experimental data (exp) [56] are shown for comparison.

In Fig. 5.14, the relative energies $(E_\nu^{n\Lambda} - E_{HF})$ of hypernuclei $^{16}_{\Lambda}$O, $^{40}_{\Lambda}$Ca, $^{48}_{\Lambda}$Ca with respect to the ground-state energies E_{HF} of the corresponding nuclei (^{16}O, ^{40}Ca, ^{48}Ca) calculated with NN + NNN interaction with $N_{max} = 6, 8, 10$, and 12 are shown. The red lines in Fig. 5.14 represent the states with negative parity, the blue lines represent the states with positive parity. In $^{40}_{\Lambda}$Ca and $^{48}_{\Lambda}$Ca, the states are coupled in multiplets. The lowest energy levels shown in Fig. 5.14 exhibit quick convergence.

In Fig. 5.15, the relative energies $(E_\nu^{n\Lambda} - E_1^{n\Lambda})$ of $^{16}_{\Lambda}$O with respect to the lowest state 0^-, calculated only with NN interaction and NN + NNN interaction, together with experimental data are presented. In calculations with NN + NNN force the relative energies of $^{16}_{\Lambda}$O are strongly quenched. The effect of NNN interaction improves the agreement of calculated energies with the experimental ones.

In Fig. 5.16, the relative energies $(E_\nu^{n\Lambda} - E_1^{n\Lambda})$ of $^{40}_{\Lambda}$Ca and $^{48}_{\Lambda}$Ca with respect to the lowest states 1^+ and 4^- are shown for calculations with only NN interaction and NN + NNN interaction. We observe similar effects as for the case of $^{16}_{\Lambda}$O. The relative distances between the energy levels and their multiplets are quenched as a result of repulsive effect of NNN interaction.

In Fig. 5.17, the relative energies $(E_\nu^{n\Lambda} - E_{HF})$ of $^{16}_{\Lambda}$O, $^{40}_{\Lambda}$Ca, and $^{48}_{\Lambda}$Ca with respect

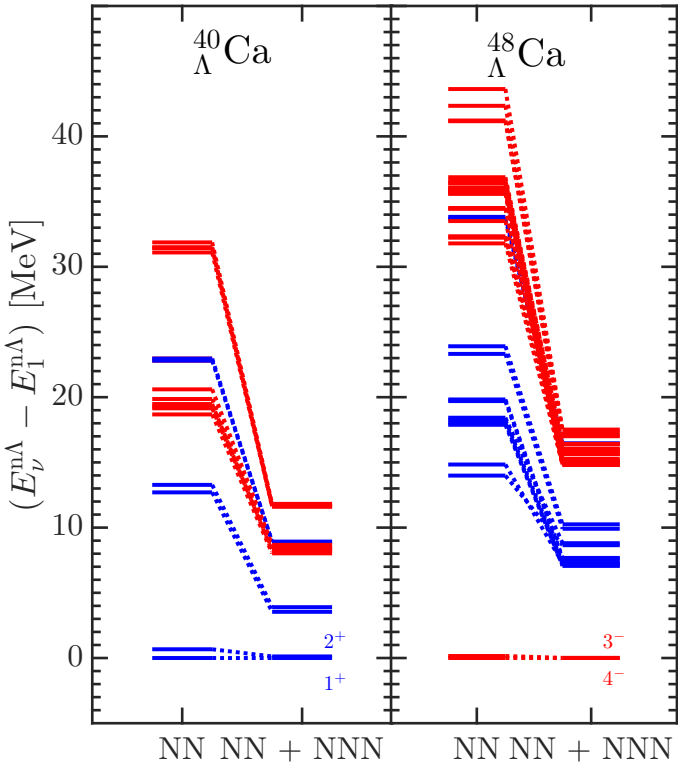

Fig. 5.16: The relative energies $(E_\nu^{n\Lambda} - E_1^{n\Lambda})$ of $^{40}_{\Lambda}$Ca and $^{48}_{\Lambda}$Ca calculated with respect to the lowest levels of $^{40}_{\Lambda}$Ca and $^{48}_{\Lambda}$Ca, respectively, with only NN interaction and with NN + NNN interaction.

to the ground-state energies $E_{\rm HF}$ calculated with NN + NNN interaction with cutoff parameter $\lambda = 550$ and 600 MeV are shown. The spectra are systematically shifted upwards in energy with higher λ. The relative distances among the levels or the multiplets depend on λ as well. However, this dependence is much smaller compared to the effect of the NNN force.

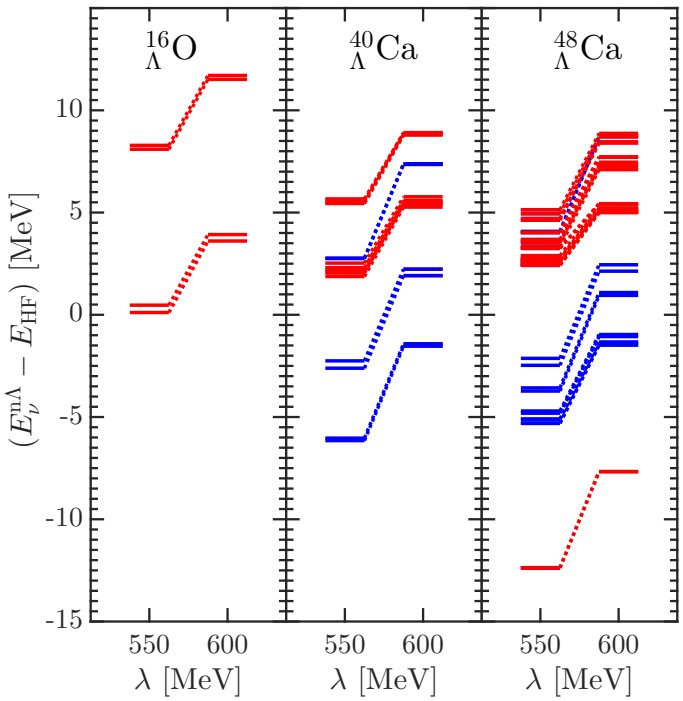

Fig. 5.17: The relative energies $(E_\nu^{n\Lambda} - E_{HF})$ of $^{16}_{\Lambda}$O, $^{40}_{\Lambda}$Ca, and $^{48}_{\Lambda}$Ca calculated for the cutoff parameter $\lambda = 550$ and 600 MeV.

Chapter 6

Conclusions

In this work, we studied the ground-state properties of doubly-magic nuclei ^{16}O, ^{40}Ca, and ^{48}Ca, and spectra of medium-mass single-Λ hypernuclei $^{17}_{\Lambda}$O, $^{41}_{\Lambda}$Ca, $^{49}_{\Lambda}$Ca, $^{16}_{\Lambda}$O, $^{40}_{\Lambda}$Ca, and $^{48}_{\Lambda}$Ca. We derived the proton-neutron-Λ (p-n-Λ) formalism of the Hartree-Fock (HF) method and the NΛ TDA method with explicit three-body NNN and ΛNN interactions. The HF method in the p-n-Λ formalism was used for calculations of the hypernuclei with one Λ particle coupled to the even-even nuclear cores – $^{17}_{\Lambda}$O, $^{41}_{\Lambda}$Ca, and $^{49}_{\Lambda}$Ca. The NΛ TDA was used for description of the hypernuclei with one Λ particle coupled to the even-odd nuclear cores – $^{16}_{\Lambda}$O, $^{40}_{\Lambda}$Ca, and $^{48}_{\Lambda}$Ca. Moreover, we laid out possible extensions of both used methods – the HF method in the p-n-Λ formalism and the NΛ TDA method – towards a more general formalism. In this formalism, the Λ particle and the NΛ TDA phonon states would be coupled with general multiphonon excitations of the nuclear core generated by the Equation of Motion Phonon Method (EMPM).

We implemented the chiral N^{2}LO$_{\text{sat}}$ NN and NNN potential as interaction among nucleons, and the ΛN-ΛN channel of the chiral LO YN potential as the two-body ΛN interaction. Although the whole theoretical formalism was derived with the three-body ΛNN force, we did not yet implement any version of the ΛNN interaction. The computer code for the HF method in the p-n-Λ formalism could run in the static or dynamic mode. In the static mode, the code solved first the HF equations for the bare nuclear core using only the NN and NNN interaction. The HF equation for the Λ particle was solved independently. The HF code in the static mode was used for the description of the nuclei ^{16}O, ^{40}Ca, and ^{48}Ca, and also as the starting point for the NΛ TDA calculations of the hypernuclei $^{16}_{\Lambda}$O, $^{40}_{\Lambda}$Ca, and $^{48}_{\Lambda}$Ca. The HF code in the dynamic mode solved all HF equations together in each iteration. The dynamic mode was used for the description of hypernuclei $^{17}_{\Lambda}$O, $^{41}_{\Lambda}$Ca, and $^{49}_{\Lambda}$Ca.

We studied the nuclei ^{16}O, ^{40}Ca, and ^{48}Ca within the HF method. We studied the

convergence of the radial density distributions, charge radii, nucleon single-particle energies, and the binding energies with respect to the size of the basis truncated by the maximal oscillator shell number $N_{\max}$. The maximal configuration space in our calculations was $N_{\max} = 12$. Calculations with larger $N_{\max}$ are currently unfeasible due to the computational complexity of generating and storing the three-nucleon matrix elements. We reached convergence for the nuclei ^{16}O and ^{40}Ca. We need to perform calculations with larger $N_{\max}$ in order to reach convergence for ^{48}Ca. We calculated radial density distributions, charge radii, nucleon single-particle energies, and binding energies of the studied nuclei. We found that the NNN force within the used potential $\mathrm{N^2LO_{sat}}$ has a very strong impact on these observables. The nuclear density distributions and charge radii calculated with only the two-nucleon NN force were too compressed. As a result, the gaps between the neutron single-particle energies were unrealistically large. The NNN force had a repulsive effect which flattened the density distributions, enhanced the charge radii, and improved the description of these observables compared to empirical data.

The single-Λ hypernuclei ${}^{17}_{\Lambda}$O, ${}^{41}_{\Lambda}$Ca, and ${}^{49}_{\Lambda}$Ca were calculated by the HF method in the p-n-Λ formalism. These hypernuclei consisted of one Λ hyperon bound to the even-even nuclear core. We studied the Λ single-particle spectra. The states with negative energies ε^{Λ} quickly reached convergence with respect to the $N_{\max}$. The states with positive ε^{Λ} required bigger configuration space for reaching the convergence. The NNN force quenched the gaps between the single-particle energies ε^{Λ}. Moreover, we found a strong dependence of the energy spectra on the cutoff parameter λ of the employed YN interaction. The Λ single-particle energy spectra of the studied hypernuclei shift systematically in energy with λ. For this reason, we compared the relative distances between Λ single-particle energies, rather than the absolute values of ε^{Λ}, to available empirical data.

The single-Λ hypernuclei ${}^{16}_{\Lambda}$O, ${}^{40}_{\Lambda}$Ca, and ${}^{48}_{\Lambda}$Ca were studied by NΛ TDA method. We studied their energy spectra. We found that the spectra of all studied hypernuclei reached convergence with respect to $N_{\max}$. The repulsive NNN force quenched the relative distances among the energy states of ${}^{16}_{\Lambda}$O, as well as among multiplets of states of ${}^{40}_{\Lambda}$Ca and ${}^{48}_{\Lambda}$Ca. We studied directly the relative distances between energy levels of ${}^{16}_{\Lambda}$O, ${}^{40}_{\Lambda}$Ca, and ${}^{48}_{\Lambda}$Ca in order to subtract the dependence of these energies on the cutoff parameter λ.

We plan to further improve our theoretical formalism. We aim to include short-range correlations into our Hamiltonian by implementing the SRG transformation of the used baryon interactions. The SRG transformation of the YN interaction produces the three-body YNN force. We plan to implement this induced three-body

interaction. It would also be desirable to implement the chiral next-to-leading order (NLO) YN interaction. We would like to address the problem of the $\Lambda - \Sigma$ mixing. Our plan is to resum the NΛ-NΣ channel of the YN interaction into the NΛ-NΛ part. Finally, we intend to study the effect of many-body correlations in hypernuclei by coupling of the Λ particle or the NΛ TDA states with the excitations of the nuclear cores. We plan to improve the theoretical formalism presented in the thesis in order to obtain more precise description of medium-mass and heavy hypernuclei directly from the realistic baryon potentials.

Appendix A

Matrix elements in izotropic harmonic oscillator basis

The basis of the izotropic harmonic oscillator (HO) is constructed from the single-particle states $|i\rangle$ which are denoted with the quantum numbers n, l, j, and m:

$$|i\rangle = |n_i l_i j_i m_i\rangle, \tag{A.1}$$

where n_i is the principal quantum number, l_i is the orbital angular momentum, j_i is the total angular momentum, and m_i is the projection of the total angular momentum. The numbers l_i, j_i and m_i satisfy the following relations

$$\left| l_i - \frac{1}{2} \right| \leq j_i \leq l_i + \frac{1}{2}, \tag{A.2a}$$

$$m_i = -j_i, -j_i + 1, \ldots, +j_i - 1, +j_i. \tag{A.2b}$$

The wave function $\psi_{n_i l_i j_i m_i}(\vec{r})$ corresponding to the state $|i\rangle$ is formulated as

$$\psi_{n_i l_i j_i m_i} = R_{n_i l_i}(r, b) \cdot \left[Y_{l_i}(\phi, \Omega) \otimes \chi_{\frac{1}{2}} \right]_{j_i m_i}. \tag{A.3}$$

The radial part $R_{n_i l_i}(r, b)$ of the wave function in (A.3) is defined as

$$R_{n_i l_i}(r, b) = b^{\frac{3}{2}} \sqrt{\frac{2n_i!}{(n_i + l_i + \frac{1}{2})!}} (br)^{l_i} L_{n_i}^{(l_i + \frac{1}{2})}(b^2 r^2) \mathrm{e}^{-\frac{b^2 r^2}{2}}, \tag{A.4}$$

where $L_{n_i}^{(l_i + \frac{1}{2})}$ is the generalized Laguerre polynomial.

Below we express the matrix elements of the kinetic energy operator, as well as all interaction terms used in the Hamiltonian (2.32) in the HO basis. The kinetic energy matrix elements $t_{ij}^{\mathrm{p,n}}$, t_{ij}^{Λ} in (2.32) are expressed as:

$$t_{ij}^{\mathrm{p,n}} = \left(1 - \frac{1}{(A-1)+\frac{M_\Lambda}{M}}\right) \left\langle n_i l_i j_i m_i \left| \frac{\widehat{\vec{P}}^2}{2M} \right| n_j l_j j_j m_j \right\rangle =$$
$$\left(1 - \frac{1}{(A-1)+\frac{M_\Lambda}{M}}\right) \left[\frac{1}{2}\hbar\omega \left(2n_i + l_i + \frac{3}{2}\right) \delta_{n_i n_j} \delta_{l_i l_j} \delta_{j_i j_j} \delta_{m_i m_j}\right.$$
$$+ \frac{1}{2}\hbar\omega \sqrt{n_i \left(n_i + l_i + \frac{1}{2}\right)} \delta_{n_i n_j+1} \delta_{l_i l_j} \delta_{j_i j_j} \delta_{m_i m_j}$$
$$\left. + \frac{1}{2}\hbar\omega \sqrt{n_j \left(n_j + l_j + \frac{1}{2}\right)} \delta_{n_i+1\, n_j} \delta_{l_i l_j} \delta_{m_i m_j}\right], \tag{A.5}$$

$$t_{ij}^{\Lambda} = \left(1 - \frac{1}{1+(A-1)\frac{M}{M_\Lambda}}\right) \left\langle n_i l_i j_i m_i \left| \frac{\widehat{\vec{P}}^2}{2M_\Lambda} \right| n_j l_j j_j m_j \right\rangle =$$
$$\left(1 - \frac{1}{1+(A-1)\frac{M}{M_\Lambda}}\right) \left[\frac{1}{2}\hbar\omega \left(2n_i + l_i + \frac{3}{2}\right) \delta_{n_i n_j} \delta_{l_i l_j} \delta_{j_i j_j} \delta_{m_i m_j}\right.$$
$$+ \frac{1}{2}\hbar\omega \sqrt{n_i \left(n_i + l_i + \frac{1}{2}\right)} \delta_{n_i n_j+1} \delta_{l_i l_j} \delta_{j_i j_j} \delta_{m_i m_j}$$
$$\left. + \frac{1}{2}\hbar\omega \sqrt{n_j \left(n_j + l_j + \frac{1}{2}\right)} \delta_{n_i+1\, n_j} \delta_{l_i l_j} \delta_{m_i m_j}\right]. \tag{A.6}$$

The matrix elements of the two-body interactions in (2.32) are defined as:

$$V_{ijkl}^{\mathrm{pp,nn}} = \left\langle ij \left| \left(\widehat{V}^{\mathrm{pp,nn}} - \frac{\widehat{\vec{P}}_1 \cdot \widehat{\vec{P}}_2}{(A-1)M + M_\Lambda} \right) \right| kl - lk \right\rangle, \tag{A.7a}$$

$$V_{ijkl}^{\mathrm{pn,p\Lambda,n\Lambda}} = \left\langle ij \left| \left(\widehat{V}^{\mathrm{pn,p\Lambda,n\Lambda}} - \frac{\widehat{\vec{P}}_1 \cdot \widehat{\vec{P}}_2}{(A-1)M + M_\Lambda} \right) \right| kl \right\rangle. \tag{A.7b}$$

The antisymmetrized matrix elements $\langle ij|\frac{\widehat{\vec{P}}_1 \cdot \widehat{\vec{P}}_2}{(A-1)M+M_\Lambda}|kl-lk\rangle$, as well as the symmetrized matrix elements $\langle ij|\frac{\widehat{\vec{P}}_1 \cdot \widehat{\vec{P}}_2}{(A-1)M+M_\Lambda}|kl\rangle$ are generated by the CENS code [57].

The matrix elements of the three-body NNN interaction in Hamiltonian (2.32)

are defined as:

$$V^{\mathrm{ppp}}_{ijklmn} = \langle ijk|\widehat{V}^{\mathrm{ppp}}|lmn - lnm + nlm - nml + mnl - mln\rangle, \tag{A.8a}$$

$$V^{\mathrm{nnn}}_{ijklmn} = \langle ijk|\widehat{V}^{\mathrm{nnn}}|lmn - lnm + nlm - nml + mnl - mln\rangle, \tag{A.8b}$$

$$V^{\mathrm{ppn}}_{ijklmn} = \langle ijk|\widehat{V}^{\mathrm{ppn}}|lmn - mln\rangle, \tag{A.8c}$$

$$V^{\mathrm{pnn}}_{ijklmn} = \langle ijk|\widehat{V}^{\mathrm{pnn}}|lmn - lnm\rangle, \tag{A.8d}$$

$$V^{\mathrm{pp}\Lambda}_{ijklmn} = \langle ijk|\widehat{V}^{\mathrm{pp}\Lambda}|lmn - mln\rangle, \tag{A.8e}$$

$$V^{\mathrm{nn}\Lambda}_{ijklmn} = \langle ijk|\widehat{V}^{\mathrm{nn}\Lambda}|lmn - mln\rangle, \tag{A.8f}$$

$$V^{\mathrm{pn}\Lambda}_{ijklmn} = \langle ijk|\widehat{V}^{\mathrm{pn}\Lambda}|lmn\rangle. \tag{A.8g}$$

The interaction matrix elements $\langle ij|\widehat{V}^{\mathrm{pp,nn}}|kl - lk\rangle$, $\langle ij|\widehat{V}^{\mathrm{pn}}|kl\rangle$, $V^{\mathrm{ppp}}_{ijklmn}, V^{\mathrm{nnn}}_{ijklmn}, V^{\mathrm{ppn}}_{ijklmn}, V^{\mathrm{pnn}}_{ijklmn}$ were provided to us by Petr Navrátil. The interaction matrix elements $\langle ij|\widehat{V}^{\mathrm{p}\Lambda}|kl\rangle, \langle ij|\widehat{V}^{\mathrm{n}\Lambda}|kl\rangle$ were provided to us by Daniel Gazda.

Appendix B

J-scheme formalism

The formalism which uses the eigenstates as defined in (A.1) is called the M-scheme. If the studied system exhibits spherical symmetry, we can develop a formalism which disregards the projections of the total angular momenta and represents the eigenstates as sets of three quantum numbers

$$|i\rangle \rightarrow (n_i l_i j_i). \tag{B.1}$$

This formalism is called the J-scheme.

The matrix elements of the kinetic operator of protons, neutrons, and the Λ particle, respectively, are transformed into the J-scheme formalism as follows

$$t^{\mathrm{p}}_{n_i l_i j_i m_i, n_j l_j j_j m_j} = t^{\mathrm{p}}_{(n_i l_i j_i),(n_j l_j j_j)} \delta_{l_i l_j} \delta_{j_i j_j} \delta_{m_i m_j}, \tag{B.2a}$$

$$t^{\mathrm{n}}_{n_i l_i j_i m_i, n_j l_j j_j m_j} = t^{\mathrm{n}}_{(n_i l_i j_i),(n_j l_j j_j)} \delta_{l_i l_j} \delta_{j_i j_j} \delta_{m_i m_j}, \tag{B.2b}$$

$$t^{\Lambda}_{n_i l_i j_i m_i, n_j l_j j_j m_j} = t^{\Lambda}_{(n_i l_i j_i),(n_j l_j j_j)} \delta_{l_i l_j} \delta_{j_i j_j} \delta_{m_i m_j}. \tag{B.2c}$$

Analogically, the matrix elements of the density matrices of protons, neutrons, and the Λ particle, respectively, read

$$\rho^{\mathrm{p}}_{n_i l_i j_i m_i, n_j l_j j_j m_j} = \rho^{\mathrm{p}}_{(n_i l_i j_i),(n_j l_j j_j)} \delta_{l_i l_j} \delta_{j_i j_j} \delta_{m_i m_j} \tag{B.3a}$$

$$\rho^{\mathrm{n}}_{n_i l_i j_i m_i, n_j l_j j_j m_j} = \rho^{\mathrm{n}}_{(n_i l_i j_i),(n_j l_j j_j)} \delta_{l_i l_j} \delta_{j_i j_j} \delta_{m_i m_j}, \tag{B.3b}$$

$$\rho^{\Lambda}_{n_i l_i j_i m_i, n_j l_j j_j m_j} = \rho^{\Lambda}_{(n_i l_i j_i),(n_j l_j j_j)} \delta_{l_i l_j} \delta_{j_i j_j} \delta_{m_i m_j}. \tag{B.3c}$$

The transformations of the matrix elements of the two-body NN and ΛN interaction operators into the J-scheme are expressed as:

$$V^{\mathrm{pp}}_{n_i l_i j_i m_i, n_j l_j j_j m_j, n_k l_k j_k m_k, n_l l_l j_l m_l} = \sum_J C^{Jm_i+m_j}_{j_i m_i, j_j m_j} C^{Jm_k+m_l}_{j_k m_k, j_l m_l} \delta_{m_k+m_l, m_i+m_j} V^{J,\mathrm{pp}}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l)}, \tag{B.4a}$$

$$V^{\mathrm{pn}}_{n_i l_i j_i m_i, n_j l_j j_j m_j, n_k l_k j_k m_k, n_l l_l j_l m_l} = \sum_J C^{Jm_i+m_j}_{j_i m_i, j_j m_j} C^{Jm_k+m_l}_{j_k m_k, j_l m_l} \delta_{m_k+m_l, m_i+m_j} V^{J,\mathrm{pn}}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l)}, \tag{B.4b}$$

$$V^{\mathrm{nn}}_{n_i l_i j_i m_i, n_j l_j j_j m_j, n_k l_k j_k m_k, n_l l_l j_l m_l} = \sum_J C^{Jm_i+m_j}_{j_i m_i, j_j m_j} C^{Jm_k+m_l}_{j_k m_k, j_l m_l} \delta_{m_k+m_l, m_i+m_j} V^{J,\mathrm{nn}}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l)}, \tag{B.4c}$$

$$V^{\mathrm{p}\Lambda}_{n_i l_i j_i m_i, n_j l_j j_j m_j, n_k l_k j_k m_k, n_l l_l j_l m_l} = \sum_J C^{Jm_i+m_j}_{j_i m_i, j_j m_j} C^{Jm_k+m_l}_{j_k m_k, j_l m_l} \delta_{m_k+m_l, m_i+m_j} V^{J,\mathrm{p}\Lambda}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l)}, \tag{B.4d}$$

$$V^{\mathrm{n}\Lambda}_{n_i l_i j_i m_i, n_j l_j j_j m_j, n_k l_k j_k m_k, n_l l_l j_l m_l} = \sum_J C^{Jm_i+m_j}_{j_i m_i, j_j m_j} C^{Jm_k+m_l}_{j_k m_k, j_l m_l} \delta_{m_k+m_l, m_i+m_j} V^{J,\mathrm{n}\Lambda}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l)}, \tag{B.4e}$$

where the symbols

$$C^{JM}_{j_i m_i, j_j m_j} = \langle j_i m_i, j_j m_j | JM \rangle,$$

represent the Clebsch-Gordan coefficients. The transformation of the matrix elements of the three-body interactions requires the following relations

$$|n_1 l_1 j_1, n_2 l_2 j_2, n_3 l_3 j_3; J_{12}, JM\rangle = \sum_{m_1 m_2 m_3} \sum_{M_{12}} C^{J_{12} M_{12}}_{j_1 m_1, j_2 m_2} C^{JM}_{J_{12} M_{12}, j_3 m_3} |n_1 l_1 j_1 m_1, n_2 l_2 j_2 m_2, n_3 l_3 j_3 m_3\rangle, \tag{B.5}$$

$$|n_1 l_1 j_1 m_1, n_2 l_2 j_2 m_2, n_3 l_3 j_3 m_3\rangle = \sum_{J_{12} J} C^{J_{12} m_1+m_2}_{j_1 m_1, j_2 m_2} C^{J m_1+m_2+m_3}_{J_{12} m_1+m_2, j_3 m_3} |n_1 l_1 j_1, n_2 l_2 j_2, n_3 l_3 j_3; J_{12}, J\ m_1{+}m_2{+}m_3\rangle. \tag{B.6}$$

Here, the angular momenta j_1 and j_2 are coupled together to J_{12}. The angular momenta J_{12} and j_3 are coupled together to J. In addition, we can introduce the isospin quantum number t and its projection m_t. The isospin t and its projection m_t of protons are $t = \frac{1}{2}, m_t = +\frac{1}{2}$, of neutrons $t = \frac{1}{2}, m_t = -\frac{1}{2}$, and of the Λ hyperon

$t = 0, m_t = 0$. Eqs. (B.5) and (B.6) can be rewritten into the JT-coupled form as

$$\begin{aligned}
&|n_1 l_1 j_1 t_1, n_2 l_2 j_2 t_2, n_3 l_3 j_3 t_3; J_{12} T_{12}, JMTM_T\rangle \\
&= \sum_{m_1 m_2 m_3} \sum_{M_{12}} \sum_{m_{t_1} m_{t_2} m_{t_3}} \sum_{M_{T_{12}}} C^{J_{12} M_{12}}_{j_1 m_1, j_2 m_2} C^{JM}_{J_{12} M_{12}, j_3 m_3} C^{T_{12} M_{T_{12}}}_{t_1 m_{t_1}, t_2 m_{t_2}} C^{TM_T}_{T_{12} M_{T_{12}}, t_3 m_{t_3}} \\
&\times |n_1 l_1 j_1 m_1 t_1 m_{t_1}, n_2 l_2 j_2 m_2 t_2 m_{t_2}, n_3 l_3 j_3 m_3 t_3 m_{t_3}\rangle, \qquad \text{(B.7)}
\end{aligned}$$

and

$$\begin{aligned}
&|n_1 l_1 j_1 m_1 t_1 m_{t_1}, n_2 l_2 j_2 m_2 t_2 m_{t_2}, n_3 l_3 j_3 m_3 t_3 m_{t_3}\rangle \\
&= \sum_{J_{12} J} \sum_{T_{12} T} C^{J_{12} m_1 + m_2}_{j_1 m_1, j_2 m_2} C^{J m_1 + m_2 + m_3}_{J_{12} m_1 + m_2, j_3 m_3} C^{T_{12} m_{t_1} + m_{t_2}}_{t_1 m_{t_1}, t_2 m_{t_2}} C^{T m_{t_1} + m_{t_2} + m_{t_3}}_{T_{12} m_{t_1} + m_{t_2}, t_3 m_{t_3}} \\
&\times |n_1 l_1 j_1 t_1, n_2 l_2 j_2 t_2, n_3 l_3 j_3 t_3; J_{12} T_{12}, J m_1 + m_2 + m_3 T m_{t_1} + m_{t_2} + m_{t_3}\rangle. \qquad \text{(B.8)}
\end{aligned}$$

In analogy to Eqs. (B.4a)-(B.4e), we can introduce the relations between the three-body interaction matrix elements in the JT-scheme and in the M-scheme:

$$\begin{aligned}
&V^{\mathrm{ppp}}_{n_i l_i j_i m_i, n_j l_j j_j m_j, n_k l_k j_k m_k, n_l l_l j_l m_l, n_m l_m j_m m_m, n_n l_n j_n m_n} \\
&= \sum_{J_{12} J'_{12}} \sum_{J} C^{J'_{12} m_i + m_j}_{j_i m_i, j_j m_j} C^{J m_i + m_j + m_k}_{J'_{12} m_i + m_j, j_k m_k} C^{J_{12} m_l + m_m}_{j_l m_l, j_m m_m} C^{J m_l + m_m + m_n}_{J_{12} m_l + m_m, j_n m_n} \\
&\quad \times V^{T'_{12}=1, T_{12}=1, T=\frac{3}{2}, J'_{12}, J_{12}, J}_{(n_i l_i j_i), (n_j l_j j_j), (n_k l_k j_k), (n_l l_l j_l), (n_m l_m j_m), (n_n l_n j_n)}, \qquad \text{(B.9)}
\end{aligned}$$

$$\begin{aligned}
&V^{\mathrm{nnn}}_{n_i l_i j_i m_i, n_j l_j j_j m_j, n_k l_k j_k m_k, n_l l_l j_l m_l, n_m l_m j_m m_m, n_n l_n j_n m_n} \\
&= \sum_{J_{12} J'_{12}} \sum_{J} C^{J'_{12} m_i + m_j}_{j_i m_i, j_j m_j} C^{J m_i + m_j + m_k}_{J'_{12} m_i + m_j, j_k m_k} C^{J_{12} m_l + m_m}_{j_l m_l, j_m m_m} C^{J m_l + m_m + m_n}_{J_{12} m_l + m_m, j_n m_n} \\
&\quad \times V^{T'_{12}=1, T_{12}=1, T=\frac{3}{2}, J'_{12}, J_{12}, J}_{(n_i l_i j_i), (n_j l_j j_j), (n_k l_k j_k), (n_l l_l j_l), (n_m l_m j_m), (n_n l_n j_n)}, \qquad \text{(B.10)}
\end{aligned}$$

$$\begin{aligned}
&V^{\mathrm{ppn}}_{n_i l_i j_i m_i, n_j l_j j_j m_j, n_k l_k j_k m_k, n_l l_l j_l m_l, n_m l_m j_m m_m, n_n l_n j_n m_n} \\
&= \sum_{J_{12} J'_{12}} \sum_{J} C^{J'_{12} m_i + m_j}_{j_i m_i, j_j m_j} C^{J m_i + m_j + m_k}_{J'_{12} m_i + m_j, j_k m_k} C^{J_{12} m_l + m_m}_{j_l m_l, j_m m_m} C^{J m_l + m_m + m_n}_{J_{12} m_l + m_m, j_n m_n} \\
&\quad \left(\frac{2}{3} V^{T'_{12}=1, T_{12}=1, T=\frac{1}{2}, J'_{12}, J_{12}, J}_{(n_i l_i j_i), (n_j l_j j_j), (n_k l_k j_k), (n_l l_l j_l), (n_m l_m j_m), (n_n l_n j_n)} \right. \\
&\quad \left. + \frac{1}{3} V^{T'_{12}=1, T_{12}=1, T=\frac{3}{2}, J'_{12}, J_{12}, J}_{(n_i l_i j_i), (n_j l_j j_j), (n_k l_k j_k), (n_l l_l j_l), (n_m l_m j_m), (n_n l_n j_n)} \right), \qquad \text{(B.11)}
\end{aligned}$$

$$
\begin{aligned}
&V^{\mathrm{pnn}}_{n_i l_i j_i m_i, n_j l_j j_j m_j, n_k l_k j_k m_k, n_l l_l j_l m_l, n_m l_m j_m m_m, n_n l_n j_n m_n} \\
&= \sum_{J_{12} J'_{12}} \sum_{J} C^{J'_{12} m_i+m_j}_{j_i m_i, j_j m_j} C^{J m_i+m_j+m_k}_{J'_{12} m_i+m_j, j_k m_k} C^{J_{12} m_l+m_m}_{j_l m_l, j_m m_m} C^{J m_l+m_m+m_n}_{J_{12} m_l+m_m, j_n m_n} \\
&\quad \times \Bigg(\frac{1}{2} V^{T'_{12}=0, T_{12}=0, T=\frac{1}{2}, J'_{12}, J_{12}, J}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l),(n_m l_m j_m),(n_n l_n j_n)} \\
&\quad + \frac{1}{6} V^{T'_{12}=1, T_{12}=1, T=\frac{1}{2}, J'_{12}, J_{12}, J}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l),(n_m l_m j_m),(n_n l_n j_n)} \\
&\quad + \frac{1}{3} V^{T'_{12}=1, T_{12}=1, T=\frac{3}{2}, J'_{12}, J_{12}, J}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l),(n_m l_m j_m),(n_n l_n j_n)} \\
&\quad + \frac{1}{2\sqrt{3}} V^{T'_{12}=1, T_{12}=0, T=\frac{1}{2}, J'_{12}, J_{12}, J}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l),(n_m l_m j_m),(n_n l_n j_n)} \\
&\quad + \frac{1}{2\sqrt{3}} V^{T'_{12}=0, T_{12}=1, T=\frac{1}{2}, J'_{12}, J_{12}, J}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l),(n_m l_m j_m),(n_n l_n j_n)} \Bigg),
\end{aligned}
\tag{B.12}
$$

$$
\begin{aligned}
&V^{\mathrm{pp}\Lambda}_{n_i l_i j_i m_i, n_j l_j j_j m_j, n_k l_k j_k m_k, n_l l_l j_l m_l, n_m l_m j_m m_m, n_n l_n j_n m_n} \\
&= \sum_{J_{12} J'_{12}} \sum_{J} C^{J'_{12} m_i+m_j}_{j_i m_i, j_j m_j} C^{J m_i+m_j+m_k}_{J'_{12} m_i+m_j, j_k m_k} C^{J_{12} m_l+m_m}_{j_l m_l, j_m m_m} C^{J m_l+m_m+m_n}_{J_{12} m_l+m_m, j_n m_n} \\
&\quad \times V^{T'_{12}=1, T_{12}=1, T=1, J'_{12}, J_{12}, J}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l),(n_m l_m j_m),(n_n l_n j_n)},
\end{aligned}
\tag{B.13}
$$

$$
\begin{aligned}
&V^{\mathrm{pn}\Lambda}_{n_i l_i j_i m_i, n_j l_j j_j m_j, n_k l_k j_k m_k, n_l l_l j_l m_l, n_m l_m j_m m_m, n_n l_n j_n m_n} \\
&= \sum_{J_{12} J'_{12}} \sum_{J} C^{J'_{12} m_i+m_j}_{j_i m_i, j_j m_j} C^{J m_i+m_j+m_k}_{J'_{12} m_i+m_j, j_k m_k} C^{J_{12} m_l+m_m}_{j_l m_l, j_m m_m} C^{J m_l+m_m+m_n}_{J_{12} m_l+m_m, j_n m_n} \\
&\quad \times \Bigg(\frac{1}{2} V^{T'_{12}=1, T_{12}=1, T=1, J'_{12}, J_{12}, J}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l),(n_m l_m j_m),(n_n l_n j_n)} \\
&\quad + \frac{1}{2} V^{T'_{12}=0, T_{12}=0, T=0, J'_{12}, J_{12}, J}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l),(n_m l_m j_m),(n_n l_n j_n)} \Bigg),
\end{aligned}
\tag{B.14}
$$

$$
\begin{aligned}
&V^{\mathrm{nn}\Lambda}_{n_i l_i j_i m_i, n_j l_j j_j m_j, n_k l_k j_k m_k, n_l l_l j_l m_l, n_m l_m j_m m_m, n_n l_n j_n m_n} \\
&= \sum_{J_{12} J'_{12}} \sum_{J} C^{J'_{12} m_i+m_j}_{j_i m_i, j_j m_j} C^{J m_i+m_j+m_k}_{J'_{12} m_i+m_j, j_k m_k} C^{J_{12} m_l+m_m}_{j_l m_l, j_m m_m} C^{J m_l+m_m+m_n}_{J_{12} m_l+m_m, j_n m_n} \\
&\quad \times V^{T'_{12}=1, T_{12}=1, T=1, J'_{12}, J_{12}, J}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l),(n_m l_m j_m),(n_n l_n j_n)}.
\end{aligned}
\tag{B.15}
$$

B.1 Hartree-Fock equations in the J-scheme formalism

In this section we present the Hartree-Fock equations in the J-scheme formalism. We substitute matrix elements of the one-body, two-body, and three-body operators transformed into the J-scheme to respective Hartree-Fock equations in the M-scheme

(2.43), (2.44), and (2.45). We obtain the corresponding HF equations for protons, neutrons, and the Λ hyperon in the J-scheme:

$$
\begin{aligned}
&t^{\mathrm{p}}_{(n_i l_i j_i),(n_j l_j j_j)}\delta_{l_i l_j}\delta_{j_i j_j}\delta_{m_i m_j} \\
&+\sum_J \sum_{\substack{n_k l_k j_k \\ n_l l_l j_l}} V^{J,\mathrm{pp}}_{(n_i l_i j_i),(n_k l_k j_k),(n_j l_j j_j),(n_l l_l j_l)} \rho^{\mathrm{p}}_{(n_l l_l j_l),(n_k l_k j_k)} \delta_{l_l l_k}\delta_{j_l j_k}\delta_{m_i m_j}\frac{2J+1}{2j_i+1} \\
&+\sum_J \sum_{\substack{n_k l_k j_k \\ n_l l_l j_l}} V^{J,\mathrm{pn}}_{(n_i l_i j_i),(n_k l_k j_k),(n_j l_j j_j),(n_l l_l j_l)} \rho^{\mathrm{n}}_{(n_l l_l j_l),(n_k l_k j_k)} \delta_{l_l l_k}\delta_{j_j j_k}\delta_{m_i m_j}\frac{2J+1}{2j_i+1} \\
&+\sum_J \sum_{\substack{n_k l_k j_k \\ n_l l_l j_l}} V^{J,\mathrm{p}\Lambda}_{(n_i l_i j_i),(n_k l_k j_k),(n_j l_j j_j),(n_l l_l j_l)} \rho^{\Lambda}_{(n_l l_l j_l),(n_k l_k j_k)} \delta_{l_l l_k}\delta_{j_l j_k}\delta_{m_i m_j}\frac{2J+1}{2j_i+1}\frac{1}{2j_k+1} \\
&+\frac{1}{2}\sum_{\substack{n_k l_k j_k \\ n_m l_m j_m \\ n_l l_l j_l \\ n_n l_n j_n}} \sum_{J_{12}J} \frac{2J+1}{2j_i+1}\delta_{j_i j_j}\delta_{l_m l_k}\delta_{j_m j_k}\delta_{l_l l_n}\delta_{j_l j_n}\delta_{m_i m_j} \\
&\times V^{T'_{12}=1,T_{12}=1,T=\frac{3}{2},J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)} \rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)} \rho^{\mathrm{p}}_{(n_n l_n j_n),(n_l l_l j_l)} \\
&+\frac{1}{2}\Bigg(\frac{2}{3}\sum_{\substack{n_k l_k j_k \\ n_m l_m j_m \\ n_l l_l j_l \\ n_n l_n j_n}} \sum_{J_{12}J} \frac{2J+1}{2j_i+1}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n} \\
&\times V^{T'_{12}=1,T_{12}=1,T=\frac{1}{2},J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)} \rho^{\mathrm{n}}_{(n_m l_m j_m),(n_k l_k j_k)} \rho^{\mathrm{n}}_{(n_n l_n j_n),(n_l l_l j_l)} \\
&+\frac{1}{3}\sum_{\substack{n_k l_k j_k \\ n_m l_m j_m \\ n_l l_l j_l \\ n_n l_n j_n}} \sum_{J_{12}J} \frac{2J+1}{2j_i+1}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n} \\
&\times V^{T'_{12}=1,T_{12}=1,T=\frac{3}{2},J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)} \rho^{\mathrm{n}}_{(n_m l_m j_m),(n_k l_k j_k)} \rho^{\mathrm{n}}_{(n_n l_n j_n),(n_l l_l j_l)}\Bigg) \\
&+\frac{1}{2}\sum_{\substack{n_k l_k j_k \\ n_m l_m j_m \\ n_l l_l j_l \\ n_n l_n j_n}} \sum_{J_{12}J} \frac{2J+1}{2j_i+1}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n} \\
&\times V^{T'_{12}=0,T_{12}=0,T=\frac{1}{2},J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)} \rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)} \rho^{\mathrm{n}}_{(n_n l_n j_n),(n_l l_l j_l)} \\
&+\frac{1}{6}\sum_{\substack{n_k l_k j_k \\ n_m l_m j_m \\ n_l l_l j_l \\ n_n l_n j_n}} \sum_{J_{12}J} \frac{2J+1}{2j_i+1}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n} \\
&\times V^{T'_{12}=1,T_{12}=1,T=\frac{1}{2},J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)} \rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)} \rho^{\mathrm{n}}_{(n_n l_n j_n),(n_l l_l j_l)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{3}\sum_{\substack{n_k l_k j_k\\ n_m l_m j_m\\ n_l l_l j_l\\ n_n l_n j_n}}\sum_{J_{12}J}\frac{2J+1}{2j_i+1}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n}\\
&\times V^{T'_{12}=1,T_{12}=1,T=\frac{3}{2},J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)}\rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)}\rho^{\mathrm{n}}_{(n_n l_n j_n),(n_l l_l j_l)}\\
&-\frac{1}{2\sqrt{3}}\sum_{\substack{n_k l_k j_k\\ n_m l_m j_m\\ n_l l_l j_l\\ n_n l_n j_n}}\sum_{J_{12}J}\frac{2J+1}{2j_i+1}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n}\\
&\times V^{T'_{12}=0,T_{12}=1,T=\frac{1}{2},J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)}\rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)}\rho^{\mathrm{n}}_{(n_n l_n j_n),(n_l l_l j_l)}\\
&-\frac{1}{2\sqrt{3}}\sum_{\substack{n_k l_k j_k\\ n_m l_m j_m\\ n_l l_l j_l\\ n_n l_n j_n}}\sum_{J_{12}J}\frac{2J+1}{2j_i+1}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n}\\
&\times V^{T'_{12}=1,T_{12}=0,T=\frac{1}{2},J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)}\rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)}\rho^{\mathrm{n}}_{(n_n l_n j_n),(n_l l_l j_l)}\\
&+\sum_{\substack{n_k l_k j_k\\ n_m l_m j_m\\ n_l l_l j_l\\ n_n l_n j_n}}\sum_{J_{12}J}\frac{2J+1}{2j_i+1}\frac{1}{2j_n+1}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n}\\
&\times V^{T'_{12}=\frac{1}{2},T_{12}=\frac{1}{2},T=1,J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)}\rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)}\rho^{\Lambda}_{(n_n l_n j_n),(n_l l_l j_l)}\\
&+\frac{1}{2}\sum_{\substack{n_k l_k j_k\\ n_m l_m j_m\\ n_l l_l j_l\\ n_n l_n j_n}}\sum_{J_{12}J}\frac{2J+1}{2j_i+1}\frac{1}{2j_n+1}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n}\\
&\times V^{T'_{12}=\frac{1}{2},T_{12}=\frac{1}{2},T=0,J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)}\rho^{\mathrm{n}}_{(n_m l_m j_m),(n_k l_k j_k)}\rho^{\Lambda}_{(n_n l_n j_n),(n_l l_l j_l)}\\
&+\frac{1}{2}\sum_{\substack{n_k l_k j_k\\ n_m l_m j_m\\ n_l l_l j_l\\ n_n l_n j_n}}\sum_{J_{12}J}\frac{2J+1}{2j_i+1}\frac{1}{2j_n+1}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n}\\
&\times V^{T'_{12}=\frac{1}{2},T_{12}=\frac{1}{2},T=1,J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)}\rho^{\mathrm{n}}_{(n_m l_m j_m),(n_k l_k j_k)}\rho^{\Lambda}_{(n_n l_n j_n),(n_l l_l j_l)}=\varepsilon^{\mathrm{p}}_i\delta_{ij},
\end{aligned}
\tag{B.16}
$$

$$
\begin{aligned}
&t^{\mathrm{n}}_{(n_i l_i j_i),(n_j l_j j_j)} \delta_{l_i l_j} \delta_{j_i j_j} \delta_{m_i m_j} \\
&+ \sum_J \sum_{\substack{n_k l_k j_k \\ n_l l_l j_l}} V^{J,\mathrm{nn}}_{(n_i l_i j_i),(n_k l_k j_k),(n_j l_j j_j),(n_l l_l j_l)} \rho^{\mathrm{n}}_{(n_l l_l j_l),(n_k l_k j_k)} \delta_{l_l l_k} \delta_{j_l j_k} \delta_{m_i m_j} \frac{2J+1}{2j_i+1} \\
&+ \sum_J \sum_{\substack{n_k l_k j_k \\ n_l l_l j_l}} V^{J,\mathrm{pn}}_{(n_i l_i j_i),(n_k l_k j_k),(n_j l_j j_j),(n_l l_l j_l)} \rho^{\mathrm{n}}_{(n_l l_l j_l),(n_k l_k j_k)} \delta_{l_l l_k} \delta_{j_j j_k} \delta_{m_i m_j} \frac{2J+1}{2j_i+1} \\
&+ \sum_J \sum_{\substack{n_k l_k j_k \\ n_l l_l j_l}} V^{J,\mathrm{n}\Lambda}_{(n_i l_i j_i),(n_k l_k j_k),(n_j l_j j_j),(n_l l_l j_l)} \rho^{\Lambda}_{(n_l l_l j_l),(n_k l_k j_k)} \delta_{l_l l_k} \delta_{j_l j_k} \delta_{m_i m_j} \frac{2J+1}{2j_i+1} \frac{1}{2j_k+1} \\
&+ \frac{1}{2} \sum_{\substack{n_k l_k j_k \\ n_m l_m j_m \\ n_l l_l j_l \\ n_n l_n j_n}} \sum_{J_{12} J} \frac{2J+1}{2j_i+1} \delta_{j_i j_j} \delta_{l_m l_k} \delta_{j_m j_k} \delta_{l_l l_n} \delta_{j_l j_n} \delta_{m_i m_j} \\
&\times V^{T'_{12}=1,T_{12}=1,T=\frac{3}{2},J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)} \rho^{\mathrm{n}}_{(n_m l_m j_m),(n_k l_k j_k)} \rho^{\mathrm{n}}_{(n_n l_n j_n),(n_l l_l j_l)} \\
&+ \frac{1}{2} \Bigg(\frac{2}{3} \sum_{\substack{n_k l_k j_k \\ n_m l_m j_m \\ n_l l_l j_l \\ n_n l_n j_n}} \sum_{J_{12} J} \frac{2J+1}{2j_i+1} \delta_{j_i j_j} \delta_{m_i m_j} \delta_{l_k l_m} \delta_{j_k j_m} \delta_{l_l l_n} \delta_{j_l j_n} \\
&\times V^{T'_{12}=1,T_{12}=1,T=\frac{1}{2},J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)} \rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)} \rho^{\mathrm{p}}_{(n_n l_n j_n),(n_l l_l j_l)} \\
&+ \frac{1}{3} \sum_{\substack{n_k l_k j_k \\ n_m l_m j_m \\ n_l l_l j_l \\ n_n l_n j_n}} \sum_{J_{12} J} \frac{2J+1}{2j_i+1} \delta_{j_i j_j} \delta_{m_i m_j} \delta_{l_k l_m} \delta_{j_k j_m} \delta_{l_l l_n} \delta_{j_l j_n} \\
&\times V^{T'_{12}=1,T_{12}=1,T=\frac{3}{2},J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)} \rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)} \rho^{\mathrm{p}}_{(n_n l_n j_n),(n_l l_l j_l)} \Bigg) \\
&+ \frac{1}{2} \sum_{\substack{n_k l_k j_k \\ n_m l_m j_m \\ n_l l_l j_l \\ n_n l_n j_n}} \sum_{J_{12} J} \frac{2J+1}{2j_i+1} \delta_{j_i j_j} \delta_{m_i m_j} \delta_{l_k l_m} \delta_{j_k j_m} \delta_{l_l l_n} \delta_{j_l j_n} \\
&\times V^{T'_{12}=0,T_{12}=0,T=\frac{1}{2},J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)} \rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)} \rho^{\mathrm{n}}_{(n_n l_n j_n),(n_l l_l j_l)} \\
&+ \frac{1}{6} \sum_{\substack{n_k l_k j_k \\ n_m l_m j_m \\ n_l l_l j_l \\ n_n l_n j_n}} \sum_{J_{12} J} \frac{2J+1}{2j_i+1} \delta_{j_i j_j} \delta_{m_i m_j} \delta_{l_k l_m} \delta_{j_k j_m} \delta_{l_l l_n} \delta_{j_l j_n} \\
&\times V^{T'_{12}=1,T_{12}=1,T=\frac{1}{2},J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)} \rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)} \rho^{\mathrm{n}}_{(n_n l_n j_n),(n_l l_l j_l)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{3}\sum_{\substack{n_k l_k j_k\\ n_m l_m j_m\\ n_l l_l j_l\\ n_n l_n j_n}}\sum_{J_{12}J}\frac{2J+1}{2j_i+1}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n}\\
&\times V^{T'_{12}=1,T_{12}=1,T=\frac{3}{2},J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)}\rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)}\rho^{\mathrm{n}}_{(n_n l_n j_n),(n_l l_l j_l)}\\
&+\frac{1}{2\sqrt{3}}\sum_{\substack{n_k l_k j_k\\ n_m l_m j_m\\ n_l l_l j_l\\ n_n l_n j_n}}\sum_{J_{12}J}\frac{2J+1}{2j_i+1}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n}\\
&\times V^{T'_{12}=0,T_{12}=1,T=\frac{1}{2},J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)}\rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)}\rho^{\mathrm{n}}_{(n_n l_n j_n),(n_l l_l j_l)}\\
&+\frac{1}{2\sqrt{3}}\sum_{\substack{n_k l_k j_k\\ n_m l_m j_m\\ n_l l_l j_l\\ n_n l_n j_n}}\sum_{J_{12}J}\frac{2J+1}{2j_i+1}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n}\\
&\times V^{T'_{12}=1,T_{12}=0,T=\frac{1}{2},J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)}\rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)}\rho^{\mathrm{n}}_{(n_n l_n j_n),(n_l l_l j_l)}\\
&+\sum_{\substack{n_k l_k j_k\\ n_m l_m j_m\\ n_l l_l j_l\\ n_n l_n j_n}}\sum_{J_{12}J}\frac{2J+1}{2j_i+1}\frac{1}{2j_n+1}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n}\\
&\times V^{T'_{12}=\frac{1}{2},T_{12}=\frac{1}{2},T=1,J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)}\rho^{\mathrm{n}}_{(n_m l_m j_m),(n_k l_k j_k)}\rho^{\Lambda}_{(n_n l_n j_n),(n_l l_l j_l)}\\
&+\frac{1}{2}\sum_{\substack{n_k l_k j_k\\ n_m l_m j_m\\ n_l l_l j_l\\ n_n l_n j_n}}\sum_{J_{12}J}\frac{2J+1}{2j_i+1}\frac{1}{2j_n+1}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n}\\
&\times V^{T'_{12}=\frac{1}{2},T_{12}=\frac{1}{2},T=0,J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)}\rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)}\rho^{\Lambda}_{(n_n l_n j_n),(n_l l_l j_l)}\\
&+\frac{1}{2}\sum_{\substack{n_k l_k j_k\\ n_m l_m j_m\\ n_l l_l j_l\\ n_n l_n j_n}}\sum_{J_{12}J}\frac{2J+1}{2j_i+1}\frac{1}{2j_n+1}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n}\\
&\times V^{T'_{12}=\frac{1}{2},T_{12}=\frac{1}{2},T=1,J_{12},J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)}\rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)}\rho^{\Lambda}_{(n_n l_n j_n),(n_l l_l j_l)}=\varepsilon^{\mathrm{n}}_i\delta_{ij},
\end{aligned}
\tag{B.17}
$$

$$
\begin{aligned}
&t^{\Lambda}_{(n_i l_i j_i),(n_j l_j j_j)}\delta_{l_i l_j}\delta_{j_i j_j}\delta_{m_i m_j} \\
&+\sum_{J}\sum_{\substack{n_k l_k j_k \\ n_l l_l j_l}} V^{J,\mathrm{p}\Lambda}_{(n_i l_i j_i),(n_k l_k j_k),(n_j l_j j_j),(n_l l_l j_l)}\rho^{\mathrm{p}}_{(n_l l_l j_l),(n_k l_k j_k)}\delta_{l_k l_l}\delta_{j_k j_l}\delta_{m_i m_j}\frac{2J+1}{2j_i+1} \\
&+\sum_{J}\sum_{\substack{n_k l_k j_k \\ n_l l_l j_l}} V^{J,\mathrm{n}\Lambda}_{(n_i l_i j_i),(n_k l_k j_k),(n_j l_j j_j),(n_l l_l j_l)}\rho^{\mathrm{n}}_{(n_l l_l j_l),(n_k l_k j_k)}\delta_{l_k l_l}\delta_{j_k j_l}\delta_{m_i m_j}\frac{2J+1}{2j_i+1} \\
&+\frac{1}{2}\sum_{\substack{n_k l_k j_k \\ n_m l_m j_m \\ n_l l_l j_l \\ n_n l_n j_n}}\sum_{J_{12}J}\frac{2J+1}{2j_i+1}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n} \\
&\times V^{T'_{12}=1,T_{12}=1,T=1,J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)}\rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)}\rho^{\mathrm{p}}_{(n_n l_n j_n),(n_l l_l j_l)} \\
&+\frac{1}{2}\sum_{\substack{n_k l_k j_k \\ n_m l_m j_m \\ n_l l_l j_l \\ n_n l_n j_n}}\sum_{J_{12}J}\frac{2J+1}{2j_i+1}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n} \\
&\times V^{T'_{12}=1,T_{12}=1,T=1,J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)}\rho^{\mathrm{n}}_{(n_m l_m j_m),(n_k l_k j_k)}\rho^{\mathrm{n}}_{(n_n l_n j_n),(n_l l_l j_l)} \\
&+\frac{1}{2}\sum_{\substack{n_k l_k j_k \\ n_m l_m j_m \\ n_l l_l j_l \\ n_n l_n j_n}}\sum_{J_{12}J}\frac{2J+1}{2j_i+1}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n} \\
&\times V^{T'_{12}=0,T_{12}=0,T=0,J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)}\rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)}\rho^{\mathrm{n}}_{(n_n l_n j_n),(n_l l_l j_l)} \\
&+\frac{1}{2}\sum_{\substack{n_k l_k j_k \\ n_m l_m j_m \\ n_l l_l j_l \\ n_n l_n j_n}}\sum_{J_{12}J}\frac{2J+1}{2j_i+1}\delta_{j_i j_j}\delta_{m_i m_j}\delta_{l_k l_m}\delta_{j_k j_m}\delta_{l_l l_n}\delta_{j_l j_n} \\
&\times V^{T'_{12}=1,T_{12}=1,T=1,J_{12},J}_{(n_k l_k j_k),(n_l l_l j_l),(n_i l_i j_i),(n_m l_m j_m),(n_n l_n j_n),(n_j l_j j_j)}\rho^{\mathrm{p}}_{(n_m l_m j_m),(n_k l_k j_k)}\rho^{\mathrm{n}}_{(n_n l_n j_n),(n_l l_l j_l)} = \varepsilon^{\Lambda}_i\delta_{ij}.
\end{aligned}
\tag{B.18}
$$

The HF energy E_{HF} corresponding to the minimal value of the energy functional (2.47) is expressed in the J-scheme as follows:

$$
\begin{aligned}
E_{\mathrm{HF}} = & \sum_{\substack{n_i l_i j_i \\ n_j l_j j_j}} t^{\mathrm{p}}_{(n_i l_i j_i),(n_j l_j j_j)} \rho^{\mathrm{p}}_{(n_j l_j j_j),(n_i l_i j_i)} (2j_i+1) \delta_{l_i l_j} \delta_{j_i j_j} \\
& + \sum_{\substack{n_i l_i j_i \\ n_j l_j j_j}} t^{\mathrm{n}}_{(n_i l_i j_i),(n_j l_j j_j)} \rho^{\mathrm{n}}_{(n_j l_j j_j),(n_i l_i j_i)} (2j_i+1) \delta_{l_i l_j} \delta_{j_i j_j} \\
& + \sum_{\substack{n_i l_i j_i \\ n_j l_j j_j}} t^{\Lambda}_{(n_i l_i j_i),(n_j l_j j_j)} \rho^{\Lambda}_{(n_j l_j j_j),(n_i l_i j_i)} \delta_{l_i l_j} \delta_{j_i j_j} \\
& + \frac{1}{2} \sum_{\substack{n_i l_i j_i \\ n_j l_j j_j \\ n_k l_k j_k \\ n_l l_l j_l}} \sum_J (2J+1) V^{J,\mathrm{pp}}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l)} \rho^{\mathrm{p}}_{(n_k l_k j_k),(n_i l_i j_i)} \rho^{\mathrm{p}}_{(n_l l_l j_l),(n_j l_j j_j)} \delta_{l_i l_k} \delta_{j_i j_k} \delta_{l_j l_l} \delta_{j_i j_l} \\
& + \frac{1}{2} \sum_{\substack{n_i l_i j_i \\ n_j l_j j_j \\ n_k l_k j_k \\ n_l l_l j_l}} \sum_J (2J+1) V^{J,\mathrm{nn}}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l)} \rho^{\mathrm{n}}_{(n_k l_k j_k),(n_i l_i j_i)} \rho^{\mathrm{n}}_{(n_l l_l j_l),(n_j l_j j_j)} \delta_{l_i l_k} \delta_{j_i j_k} \delta_{l_j l_l} \delta_{j_i j_l} \\
& + \sum_{\substack{n_i l_i j_i \\ n_j l_j j_j \\ n_k l_k j_k \\ n_l l_l j_l}} \sum_J (2J+1) V^{J,\mathrm{pn}}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l)} \rho^{\mathrm{p}}_{(n_k l_k j_k),(n_i l_i j_i)} \rho^{\mathrm{n}}_{(n_l l_l j_l),(n_j l_j j_j)} \delta_{l_i l_k} \delta_{j_i j_k} \delta_{l_j l_l} \delta_{j_i j_l} \\
& + \sum_{\substack{n_i l_i j_i \\ n_j l_j j_j \\ n_k l_k j_k \\ n_l l_l j_l}} \sum_J \frac{2J+1}{2j_j+1} V^{J,\mathrm{p}\Lambda}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l)} \rho^{\mathrm{p}}_{(n_k l_k j_k),(n_i l_i j_i)} \rho^{\Lambda}_{(n_l l_l j_l),(n_j l_j j_j)} \delta_{l_i l_k} \delta_{j_i j_k} \delta_{l_j l_l} \delta_{j_i j_l} \\
& + \sum_{\substack{n_i l_i j_i \\ n_j l_j j_j \\ n_k l_k j_k \\ n_l l_l j_l}} \sum_J \frac{2J+1}{2j_j+1} V^{J,\mathrm{n}\Lambda}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l)} \rho^{\mathrm{n}}_{(n_k l_k j_k),(n_i l_i j_i)} \rho^{\Lambda}_{(n_l l_l j_l),(n_j l_j j_j)} \delta_{l_i l_k} \delta_{j_i j_k} \delta_{l_j l_l} \delta_{j_i j_l} \\
& + \frac{1}{6} \sum_{\substack{n_i l_i j_i \\ n_j l_j j_j \\ n_k l_k j_k \\ n_l l_l j_l \\ n_m l_m j_m \\ n_n l_n j_n}} \sum_{J,J_{12}} (2J_{12}+1)(2J+1) V^{T'_{12}=1,T_{12}=1,T=\frac{3}{2},J_{12},J_{12},J}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l),(n_m l_m j_m),(n_n l_n j_n)} \\
& \times \rho^{\mathrm{p}}_{(n_l l_l j_l),(n_i l_i j_i)} \rho^{\mathrm{p}}_{(n_m l_m j_m),(n_j l_j j_j)} \rho^{\mathrm{p}}_{(n_n l_n j_n),(n_k l_k j_k)} \delta_{l_i l_l} \delta_{j_i j_l} \delta_{l_j l_m} \delta_{j_j j_m} \delta_{l_k l_n} \delta_{j_k j_m} \\
& + \frac{1}{6} \sum_{\substack{n_i l_i j_i \\ n_j l_j j_j \\ n_k l_k j_k \\ n_l l_l j_l \\ n_m l_m j_m \\ n_n l_n j_n}} \sum_{J,J_{12}} (2J_{12}+1)(2J+1) V^{T'_{12}=1,T_{12}=1,T=\frac{3}{2},J_{12},J_{12},J}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l),(n_m l_m j_m),(n_n l_n j_n)} \\
& \times \rho^{\mathrm{n}}_{(n_l l_l j_l),(n_i l_i j_i)} \rho^{\mathrm{n}}_{(n_m l_m j_m),(n_j l_j j_j)} \rho^{\mathrm{n}}_{(n_n l_n j_n),(n_k l_k j_k)} \delta_{l_i l_l} \delta_{j_i j_l} \delta_{l_j l_m} \delta_{j_j j_m} \delta_{l_k l_n} \delta_{j_k j_m}
\end{aligned}
$$

$$
+\frac{1}{2}\sum_{\substack{n_il_ij_i\\ n_jl_jj_j\\ n_kl_kj_k\\ n_ll_lj_l\\ n_ml_mj_m\\ n_nl_nj_n}}\sum_{J,J_{12}}\Bigg[\frac{2}{3}(2J_{12}+1)(2J+1)V^{T'_{12}=1,T_{12}=1,T=\frac{1}{2},J_{12},J_{12},J}_{(n_il_ij_i),(n_jl_jj_j),(n_kl_kj_k),(n_ll_lj_l),(n_ml_mj_m),(n_nl_nj_n)}
$$

$$
\times\rho^{\mathrm{p}}_{(n_ll_lj_l),(n_il_ij_i)}\rho^{\mathrm{p}}_{(n_ml_mj_m),(n_jl_jj_j)}\rho^{\mathrm{n}}_{(n_nl_nj_n),(n_kl_kj_k)}\delta_{l_il_l}\delta_{j_ij_j}\delta_{l_jl_m}\delta_{j_jj_m}\delta_{l_kl_n}\delta_{j_kj_n}
$$

$$
+\frac{1}{3}(2J_{12}+1)(2J+1)V^{T'_{12}=1,T_{12}=1,T=\frac{3}{2},J_{12},J_{12},J}_{(n_il_ij_i),(n_jl_jj_j),(n_kl_kj_k),(n_ll_lj_l),(n_ml_mj_m),(n_nl_nj_n)}
$$

$$
\times\rho^{\mathrm{p}}_{(n_ll_lj_l),(n_il_ij_i)}\rho^{\mathrm{p}}_{(n_ml_mj_m),(n_jl_jj_j)}\rho^{\mathrm{n}}_{(n_nl_nj_n),(n_kl_kj_k)}\delta_{l_il_l}\delta_{j_ij_j}\delta_{l_jl_m}\delta_{j_jj_m}\delta_{l_kl_n}\delta_{j_kj_n}\Bigg]
$$

$$
+\frac{1}{2}\sum_{\substack{n_il_ij_i\\ n_jl_jj_j\\ n_kl_kj_k\\ n_ll_lj_l\\ n_ml_mj_m\\ n_nl_nj_n}}\sum_{J,J_{12}}\Bigg[\frac{2}{3}(2J_{12}+1)(2J+1)V^{T'_{12}=1,T_{12}=1,T=\frac{1}{2},J_{12},J_{12},J}_{(n_il_ij_i),(n_jl_jj_j),(n_kl_kj_k),(n_ll_lj_l),(n_ml_mj_m),(n_nl_nj_n)}
$$

$$
\times\rho^{\mathrm{n}}_{(n_ll_lj_l),(n_il_ij_i)}\rho^{\mathrm{n}}_{(n_ml_mj_m),(n_jl_jj_j)}\rho^{\mathrm{p}}_{(n_nl_nj_n),(n_kl_kj_k)}\delta_{l_il_l}\delta_{j_ij_j}\delta_{l_jl_m}\delta_{j_jj_m}\delta_{l_kl_n}\delta_{j_kj_n}
$$

$$
+\frac{1}{3}(2J_{12}+1)(2J+1)V^{T'_{12}=1,T_{12}=1,T=\frac{3}{2},J_{12},J_{12},J}_{(n_il_ij_i),(n_jl_jj_j),(n_kl_kj_k),(n_ll_lj_l),(n_ml_mj_m),(n_nl_nj_n)}
$$

$$
\times\rho^{\mathrm{n}}_{(n_ll_lj_l),(n_il_ij_i)}\rho^{\mathrm{n}}_{(n_ml_mj_m),(n_jl_jj_j)}\rho^{\mathrm{p}}_{(n_nl_nj_n),(n_kl_kj_k)}\delta_{l_il_l}\delta_{j_ij_j}\delta_{l_jl_m}\delta_{j_jj_m}\delta_{l_kl_n}\delta_{j_kj_n}\Bigg]
$$

$$
+\frac{1}{2}\sum_{\substack{n_il_ij_i\\ n_jl_jj_j\\ n_kl_kj_k\\ n_ll_lj_l\\ n_ml_mj_m\\ n_nl_nj_n}}\sum_{J,J_{12}}\frac{(2J_{12}+1)(2J+1)}{(2j_k+1)}V^{T'_{12}=1,T_{12}=1,T=1,J_{12},J_{12},J}_{(n_il_ij_i),(n_jl_jj_j),(n_kl_kj_k),(n_ll_lj_l),(n_ml_mj_m),(n_nl_nj_n)}
$$

$$
\times\rho^{\mathrm{p}}_{(n_ll_lj_l),(n_il_ij_i)}\rho^{\mathrm{p}}_{(n_ml_mj_m),(n_jl_jj_j)}\rho^{\Lambda}_{(n_nl_nj_n),(n_kl_kj_k)}\delta_{l_il_l}\delta_{j_ij_l}\delta_{l_jl_m}\delta_{j_jj_m}\delta_{l_kl_n}\delta_{j_kj_n}
$$

$$
+\frac{1}{2}\sum_{\substack{n_il_ij_i\\ n_jl_jj_j\\ n_kl_kj_k\\ n_ll_lj_l\\ n_ml_mj_m\\ n_nl_nj_n}}\sum_{J,J_{12}}\frac{(2J_{12}+1)(2J+1)}{(2j_k+1)}V^{T'_{12}=1,T_{12}=1,T=1,J_{12},J_{12},J}_{(n_il_ij_i),(n_jl_jj_j),(n_kl_kj_k),(n_ll_lj_l),(n_ml_mj_m),(n_nl_nj_n)}
$$

$$
\times\rho^{\mathrm{n}}_{(n_ll_lj_l),(n_il_ij_i)}\rho^{\mathrm{n}}_{(n_ml_mj_m),(n_jl_jj_j)}\rho^{\Lambda}_{(n_nl_nj_n),(n_kl_kj_k)}\delta_{l_il_l}\delta_{j_ij_l}\delta_{l_jl_m}\delta_{j_jj_m}\delta_{l_kl_n}\delta_{j_kj_n}
$$

$$
+\frac{1}{2}\sum_{\substack{n_il_ij_i\\ n_jl_jj_j\\ n_kl_kj_k\\ n_ll_lj_l\\ n_ml_mj_m\\ n_nl_nj_n}}\sum_{J,J_{12}}\Bigg[\frac{1}{2}\frac{(2J_{12}+1)(2J+1)}{(2j_j+1)}V^{T'_{12}=\frac{1}{2},T_{12}=\frac{1}{2},T=0,J_{12},J_{12},J}_{(n_il_ij_i),(n_jl_jj_j),(n_kl_kj_k),(n_ll_lj_l),(n_ml_mj_m),(n_nl_nj_n)}
$$

$$
\times\rho^{\mathrm{p}}_{(n_ll_lj_l),(n_il_ij_i)}\rho^{\Lambda}_{(n_ml_mj_m),(n_jl_jj_j)}\rho^{\mathrm{n}}_{(n_nl_nj_n),(n_kl_kj_k)}\delta_{l_il_l}\delta_{j_ij_j}\delta_{l_jl_m}\delta_{j_jj_m}\delta_{l_kl_n}\delta_{j_kj_n}
$$

$$
+\frac{1}{2}\frac{(2J_{12}+1)(2J+1)}{2j_j+1}V^{T'_{12}=\frac{1}{2},T_{12}=\frac{1}{2},T=1,J_{12},J_{12},J}_{(n_il_ij_i),(n_jl_jj_j),(n_kl_kj_k),(n_ll_lj_l),(n_ml_mj_m),(n_nl_nj_n)}
$$

$$
\times\rho^{\mathrm{p}}_{(n_ll_lj_l),(n_il_ij_i)}\rho^{\Lambda}_{(n_ml_mj_m),(n_jl_jj_j)}\rho^{\mathrm{n}}_{(n_nl_nj_n),(n_kl_kj_k)}\delta_{l_il_l}\delta_{j_ij_j}\delta_{l_jl_m}\delta_{j_lj_m}\delta_{l_kl_n}\delta_{j_kj_n}\Bigg]. \qquad \text{(B.19)}
$$

B.2 Tamm-Dancoff equations in the J-scheme formalism

The coefficients $C_{ph}^{\mu,\mathrm{p}}, C_{ph}^{\mu,\mathrm{n}}, r_{ph}^{\mu,\mathrm{p}\Lambda}$, and $r_{ph}^{\mu,\mathrm{n}\Lambda}$ in the TDA and NΛ TDA Eqs. (3.11), (3.19a), and (3.19b) are transformed into the J-scheme as:

$$C^{\mu,\mathrm{p}}_{n_p l_p j_p m_p, n_h l_h j_h m_h} = \sum_J (-1)^{j_h+m_h} C^{Jm_p-m_h}_{j_p m_p, j_h-m_h} C^{\mu,\mathrm{p},J}_{(n_p l_p j_p),(n_h l_h j_h)}, \tag{B.20a}$$

$$C^{\mu,\mathrm{n}}_{n_p l_p j_p m_p, n_h l_h j_h m_h} = \sum_J (-1)^{j_h+m_h} C^{Jm_p-m_h}_{j_p m_p, j_h-m_h} C^{\mu,\mathrm{n},J}_{(n_p l_p j_p),(n_h l_h j_h)}, \tag{B.20b}$$

$$r^{\mu,\mathrm{p}\Lambda}_{n_p l_p j_p m_p, n_h l_h j_h m_h} = \sum_J (-1)^{j_h+m_h} C^{Jm_p-m_h}_{j_p m_p, j_h-m_h} r^{\mu,\mathrm{p}\Lambda,J}_{(n_p l_p j_p),(n_h l_h j_h)}, \tag{B.20c}$$

$$r^{\mu,\mathrm{n}\Lambda}_{n_p l_p j_p m_p, n_h l_h j_h m_h} = \sum_J (-1)^{j_h+m_h} C^{Jm_p-m_h}_{j_p m_p, j_h-m_h} r^{\mu,\mathrm{n}\Lambda,J}_{(n_p l_p j_p),(n_h l_h j_h)}. \tag{B.20d}$$

The symbols $C^{Jm_p-m_h}_{j_p m_p, j_h-m_h}$ represent the Clebsch-Gordan coefficients. The TDA Eq. (3.11) is expressed in the J-scheme as:

$$\sum_{\substack{n_p l_p j_p \\ n_h l_h j_h}} \begin{pmatrix} \mathbb{A}^{\mathrm{pp},J'}_{\substack{(n_{p'} l_{p'} j_{p'}),(n_h l_h j_h) \\ (n_{h'} l_{h'} j_{h'}),(n_p l_p j_p)}} & \mathbb{A}^{\mathrm{pn},J'}_{\substack{(n_{p'} l_{p'} j_{p'}),(n_h l_h j_h) \\ (n_{h'} l_{h'} j_{h'}),(n_p l_p j_p)}} \\ \mathbb{A}^{\mathrm{pn},J'}_{\substack{(n_h l_h j_h),(n_{p'} l_{p'} j_{p'}) \\ (n_p l_p j_p),(n_{h'} l_{h'} j_{h'})}} & \mathbb{A}^{\mathrm{nn},J'}_{\substack{(n_{p'} l_{p'} j_{p'}),(n_h l_h j_h) \\ (n_{h'} l_{h'} j_{h'}),(n_p l_p j_p)}} \end{pmatrix} \begin{pmatrix} C^{\nu,\mathrm{p},J'}_{(n_p l_p j_p),(n_h l_h j_h)} \\ C^{\nu,\mathrm{n},J'}_{(n_p l_p j_p),(n_h l_h j_h)} \end{pmatrix}$$
$$= (E_\nu - E_{\mathrm{HF}}) \begin{pmatrix} C^{\nu,\mathrm{p},J'}_{(n_{p'} l_{p'} j_{p'}),(n_{h'} l_{h'} j_{h'})} \\ C^{\nu,\mathrm{n},J'}_{(n_{p'} l_{p'} j_{p'}),(n_{h'} l_{h'} j_{h'})} \end{pmatrix}. \tag{B.21}$$

The elements $\mathbb{A}^{\mathrm{pp},J'}_{\substack{(n_{p'}l_{p'}j_{p'}),(n_hl_hj_h)\\(n_{h'}l_{h'}j_{h'}),(n_pl_pj_p)}}$, $\mathbb{A}^{\mathrm{pn},J'}_{\substack{(n_{p'}l_{p'}j_{p'}),(n_hl_hj_h)\\(n_{h'}l_{h'}j_{h'}),(n_pl_pj_p)}}$, $\mathbb{A}^{\mathrm{pn},J'}_{\substack{(n_hl_hj_h),(n_{p'}l_{p'}j_{p'})\\(n_pl_pj_p),(n_{h'}l_{h'}j_{h'})}}$ and $\mathbb{A}^{\mathrm{nn},J'}_{\substack{(n_{p'}l_{p'}j_{p'}),(n_hl_hj_h)\\(n_{h'}l_{h'}j_{h'}),(n_pl_pj_p)}}$ in Eq. (B.21) are defined as:

$$\mathbb{A}^{\mathrm{pp},J'}_{\substack{(n_{p'}l_{p'}j_{p'}),(n_hl_hj_h)\\(n_{h'}l_{h'}j_{h'}),(n_pl_pj_p)}} = (\varepsilon^{\mathrm{p}}_{n_pl_pj_p} - \varepsilon^{\mathrm{p}}_{n_hl_hj_h})\delta_{n_pn_{p'}}\delta_{l_pl_{p'}}\delta_{j_pj_{p'}}\delta_{n_hn_{h'}}\delta_{l_hl_{h'}}\delta_{j_hj_{h'}}$$
$$+\sum_J(-1)^{J-j_p-j_{h'}}(2J+1)\begin{Bmatrix} j_{p'} & j_h & J\\ j_p & j_{h'} & J'\end{Bmatrix}\overline{V}^{J',\mathrm{pp,gen}}_{\substack{(n_{p'}l_{p'}j_{p'}),(n_hl_hj_h)\\(n_{h'}l_{h'}j_{h'}),(n_pl_pj_p)}}, \tag{B.22a}$$

$$\mathbb{A}^{\mathrm{pn},J'}_{\substack{(n_{p'}l_{p'}j_{p'}),(n_hl_hj_h)\\(n_{h'}l_{h'}j_{h'}),(n_pl_pj_p)}} = \sum_J(-1)^{J-j_p-j_{h'}}(2J+1)\begin{Bmatrix} j_{p'} & j_h & J\\ j_p & j_{h'} & J'\end{Bmatrix}\overline{V}^{J',\mathrm{pn,gen}}_{\substack{(n_{p'}l_{p'}j_{p'}),(n_hl_hj_h)\\(n_{h'}l_{h'}j_{h'}),(n_pl_pj_p)}}, \tag{B.22b}$$

$$\mathbb{A}^{\mathrm{pn},J'}_{\substack{(n_hl_hj_h),(n_{p'}l_{p'}j_{p'})\\(n_pl_pj_p),(n_{h'}l_{h'}j_{h'})}} = \sum_J(-1)^{J-j_{p'}-j_h}(2J+1)\begin{Bmatrix} j_h & j_{p'} & J\\ j_{h'} & j_p & J'\end{Bmatrix}\overline{V}^{J',\mathrm{pn,gen}}_{\substack{(n_hl_hj_h),(n_{p'}l_{p'}j_{p'})\\(n_pl_pj_p),(n_{h'}l_{h'}j_{h'})}}, \tag{B.22c}$$

$$\mathbb{A}^{\mathrm{nn},J'}_{\substack{(n_{p'}l_{p'}j_{p'}),(n_hl_hj_h)\\(n_{h'}l_{h'}j_{h'}),(n_pl_pj_p)}} = (\varepsilon^{\mathrm{n}}_{n_pl_pj_p} - \varepsilon^{\mathrm{n}}_{n_hl_hj_h})\delta_{n_pn_{p'}}\delta_{l_pl_{p'}}\delta_{j_pj_{p'}}\delta_{n_hn_{h'}}\delta_{l_hl_{h'}}\delta_{j_hj_{h'}}$$
$$+\sum_J(-1)^{J-j_p-j_{h'}}(2J+1)\begin{Bmatrix} j_{p'} & j_h & J\\ j_p & j_{h'} & J'\end{Bmatrix}\overline{V}^{J',\mathrm{nn,gen}}_{\substack{(n_{p'}l_{p'}j_{p'}),(n_hl_hj_h)\\(n_{h'}l_{h'}j_{h'}),(n_pl_pj_p)}}. \tag{B.22d}$$

The NΛ TDA Eqs. (3.19a), (3.19b) are in the J-scheme expressed as:

$$\sum_{\substack{n_pl_pj_p\\n_hl_hj_h}}\Bigg[(\varepsilon^{\Lambda}_{(n_pl_pj_p)} - \varepsilon^{\mathrm{p}}_{(n_hl_hj_h)})\delta_{n_pn_{p'}}\delta_{l_pl_{p'}}\delta_{j_pj_{p'}}\delta_{n_hn_{h'}}\delta_{l_hl_{h'}}\delta_{j_hj_{h'}}$$
$$+\sum_J(-1)^{j_p+j_{p'}+j_h+j_{h'}+1}(2J+1)\begin{Bmatrix} j_h & j_{p'} & J\\ j_{h'} & j_p & J'\end{Bmatrix}$$
$$\times\overline{V}^{J',\mathrm{p\Lambda,gen}}_{\substack{(n_hl_hj_h),(n_{p'}l_{p'}j_{p'})\\(n_{h'}l_{h'}j_{h'}),(n_pl_pj_p)}}\Bigg]r^{\nu,\mathrm{p\Lambda},J'}_{(n_pl_pj_p),(n_hl_hj_h)} = (E^{\mathrm{p}\Lambda}_{\nu} - E_{\mathrm{HF}})r^{\nu,\mathrm{p\Lambda},J'}_{(n_{p'}l_{p'}j_{p'}),(n_{h'}l_{h'}j_{h'})}, \tag{B.23}$$

$$\sum_{\substack{n_pl_pj_p\\n_hl_hj_h}}\Bigg[(\varepsilon^{\Lambda}_{(n_pl_pj_p)} - \varepsilon^{\mathrm{n}}_{(n_hl_hj_h)})\delta_{n_pn_{p'}}\delta_{l_pl_{p'}}\delta_{j_pj_{p'}}\delta_{n_hn_{h'}}\delta_{l_hl_{h'}}\delta_{j_hj_{h'}}$$
$$+\sum_J(-1)^{j_p+j_{p'}+j_h+j_{h'}+1}(2J+1)\begin{Bmatrix} j_h & j_{p'} & J\\ j_{h'} & j_p & J'\end{Bmatrix}$$
$$\times\overline{V}^{J',\mathrm{n\Lambda,gen}}_{\substack{(n_hl_hj_h),(n_{p'}l_{p'}j_{p'})\\(n_{h'}l_{h'}j_{h'}),(n_pl_pj_p)}}\Bigg]r^{\nu,\mathrm{n\Lambda},J'}_{(n_pl_pj_p),(n_hl_hj_h)} = (E^{\mathrm{p}\Lambda}_{\nu} - E_{\mathrm{HF}})r^{\nu,\mathrm{n\Lambda},J'}_{(n_{p'}l_{p'}j_{p'}),(n_{h'}l_{h'}j_{h'})}. \tag{B.24}$$

The expressions $\begin{Bmatrix} j_h & j_{p'} & J\\ j_{h'} & j_p & J'\end{Bmatrix}$ represent the Wigner 6j-symbols.

The generalized matrix elements are expressed in the J-scheme as:

$$
\begin{aligned}
&\overline{V}^{J',\mathrm{pp,gen}}_{\substack{(n_i l_i j_i),(n_j l_j j_j)\\(n_k l_k j_k),(n_l l_l j_l)}} = \overline{V}^{J',\mathrm{pp}}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l)} \\
&+ \sum_J \sum_{\substack{n_m l_m j_m \\ n_n l_n j_n}} \frac{2J+1}{2J'+1} \\
&\Big[\overline{V}^{T'_{12}=1,T_{12}=1,T=\frac{3}{2},J',J',J}_{(n_i l_i j_i),(n_j l_j j_j),(n_m l_m j_m),(n_k l_k j_k),(n_l l_l j_l),(n_n l_n j_n)} \overline{\rho}^{\mathrm{p}}_{(n_n l_n j_n),(n_m l_m j_m)} \\
&+ \frac{1}{3} \overline{V}^{T'_{12}=1,T_{12}=1,T=\frac{3}{2},J',J',J}_{(n_i l_i j_i),(n_j l_j j_j),(n_m l_m j_m),(n_k l_k j_k),(n_l l_l j_l),(n_n l_n j_n)} \overline{\rho}^{\mathrm{n}}_{(n_n l_n j_n),(n_m l_m j_m)} \\
&+ \frac{2}{3} \overline{V}^{T'_{12}=1,T_{12}=1,T=\frac{1}{2},J',J',J}_{(n_i l_i j_i),(n_j l_j j_j),(n_m l_m j_m),(n_k l_k j_k),(n_l l_l j_l),(n_n l_n j_n)} \overline{\rho}^{\mathrm{n}}_{(n_n l_n j_n),(n_m l_m j_m)} \\
&+ \overline{V}^{T'_{12}=1,T_{12}=1,T=1,J',J',J}_{(n_i l_i j_i),(n_j l_j j_j),(n_m l_m j_m),(n_k l_k j_k),(n_l l_l j_l),(n_n l_n j_n)} \overline{\rho}^{\Lambda}_{(n_n l_n j_n),(n_m l_m j_m)} \Big] .
\end{aligned}
\tag{B.25}
$$

$$
\begin{aligned}
&\overline{V}^{J',\mathrm{nn,gen}}_{\substack{(n_i l_i j_i),(n_j l_j j_j)\\(n_k l_k j_k),(n_l l_l j_l)}} = \overline{V}^{J',\mathrm{nn}}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l)} \\
&+ \sum_J \sum_{\substack{n_m l_m j_m \\ n_n l_n j_n}} \frac{2J+1}{2J'+1} \\
&\Big[\overline{V}^{T'_{12}=1,T_{12}=1,T=\frac{3}{2},J',J',J}_{(n_i l_i j_i),(n_j l_j j_j),(n_m l_m j_m),(n_k l_k j_k),(n_l l_l j_l),(n_n l_n j_n)} \overline{\rho}^{\mathrm{n}}_{(n_n l_n j_n),(n_m l_m j_m)} \\
&+ \frac{1}{3} \overline{V}^{T'_{12}=1,T_{12}=1,T=\frac{3}{2},J',J',J}_{(n_i l_i j_i),(n_j l_j j_j),(n_m l_m j_m),(n_k l_k j_k),(n_l l_l j_l),(n_n l_n j_n)} \overline{\rho}^{\mathrm{p}}_{(n_n l_n j_n),(n_m l_m j_m)} \\
&+ \frac{2}{3} \overline{V}^{T'_{12}=1,T_{12}=1,T=\frac{1}{2},J',J',J}_{(n_i l_i j_i),(n_j l_j j_j),(n_m l_m j_m),(n_k l_k j_k),(n_l l_l j_l),(n_n l_n j_n)} \overline{\rho}^{\mathrm{p}}_{(n_n l_n j_n),(n_m l_m j_m)} \\
&+ \overline{V}^{T'_{12}=1,T_{12}=1,T=1,J',J',J}_{(n_i l_i j_i),(n_j l_j j_j),(n_m l_m j_m),(n_k l_k j_k),(n_l l_l j_l),(n_n l_n j_n)} \overline{\rho}^{\Lambda}_{(n_n l_n j_n),(n_m l_m j_m)} \Big] .
\end{aligned}
\tag{B.26}
$$

$$
\begin{aligned}
&\overline{V}^{J',\mathrm{pn,gen}}_{\substack{(n_i l_i j_i),(n_j l_j j_j)\\(n_k l_k j_k),(n_l l_l j_l)}} = \overline{V}^{J',\mathrm{pn}}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l)} \\
&+ \sum_J \sum_{\substack{n_m l_m j_m \\ n_n l_n j_n}} \frac{2J+1}{2J'+1} \\
&\Bigg[\frac{1}{2} \overline{V}^{T'_{12}=0,T_{12}=0,T=\frac{1}{2},J',J',J}_{(n_i l_i j_i),(n_j l_j j_j),(n_m l_m j_m),(n_k l_k j_k),(n_l l_l j_l),(n_n l_n j_n)} \overline{\rho}^{\mathrm{p}}_{(n_n l_n j_n),(n_m l_m j_m)} \\
&+ \frac{1}{6} \overline{V}^{T'_{12}=1,T_{12}=1,T=\frac{1}{2},J',J',J}_{(n_i l_i j_i),(n_j l_j j_j),(n_m l_m j_m),(n_k l_k j_k),(n_l l_l j_l),(n_n l_n j_n)} \overline{\rho}^{\mathrm{p}}_{(n_n l_n j_n),(n_m l_m j_m)} \\
&+ \frac{1}{3} \overline{V}^{T'_{12}=1,T_{12}=1,T=\frac{3}{2},J',J',J}_{(n_i l_i j_i),(n_j l_j j_j),(n_m l_m j_m),(n_k l_k j_k),(n_l l_l j_l),(n_n l_n j_n)} \overline{\rho}^{\mathrm{p}}_{(n_n l_n j_n),(n_m l_m j_m)} \\
&- \frac{1}{2\sqrt{3}} \overline{V}^{T'_{12}=0,T_{12}=1,T=\frac{1}{2},J',J',J}_{(n_i l_i j_i),(n_j l_j j_j),(n_m l_m j_m),(n_k l_k j_k),(n_l l_l j_l),(n_n l_n j_n)} \overline{\rho}^{p}_{(n_n l_n j_n),(n_m l_m j_m)} \\
&- \frac{1}{2\sqrt{3}} \overline{V}^{T'_{12}=1,T_{12}=0,T=\frac{1}{2},J',J',J}_{(n_i l_i j_i),(n_j l_j j_j),(n_m l_m j_m),(n_k l_k j_k),(n_l l_l j_l),(n_n l_n j_n)} \overline{\rho}^{p}_{(n_n l_n j_n),(n_m l_m j_m)} \\
&+ \frac{1}{3} \overline{V}^{T'_{12}=1,T_{12}=1,T=\frac{3}{2},J',J',J}_{(n_i l_i j_i),(n_j l_j j_j),(n_m l_m j_m),(n_k l_k j_k),(n_l l_l j_l),(n_n l_n j_n)} \overline{\rho}^{n}_{(n_n l_n j_n),(n_m l_m j_m)} \\
&+ \frac{2}{3} \overline{V}^{T'_{12}=1,T_{12}=1,T=\frac{1}{2},J',J',J}_{(n_i l_i j_i),(n_j l_j j_j),(n_m l_m j_m),(n_k l_k j_k),(n_l l_l j_l),(n_n l_n j_n)} \overline{\rho}^{n}_{(n_n l_n j_n),(n_m l_m j_m)} \\
&+ \frac{1}{2} \overline{V}^{T'_{12}=1,T_{12}=1,T=1,J',J',J}_{(n_i l_i j_i),(n_j l_j j_j),(n_m l_m j_m),(n_k l_k j_k),(n_l l_l j_l),(n_n l_n j_n)} \overline{\rho}^{\Lambda}_{(n_n l_n j_n),(n_m l_m j_m)} \\
&+ \frac{1}{2} \overline{V}^{T'_{12}=0,T_{12}=0,T=0,J',J',J}_{(n_i l_i j_i),(n_j l_j j_j),(n_m l_m j_m),(n_k l_k j_k),(n_l l_l j_l),(n_n l_n j_n)} \overline{\rho}^{\Lambda}_{(n_n l_n j_n),(n_m l_m j_m)} \Bigg] .
\end{aligned}
\tag{B.27}
$$

$$
\begin{aligned}
&\overline{V}^{J',\mathrm{p}\Lambda,\mathrm{gen}}_{\substack{(n_i l_i j_i),(n_j l_j j_j)\\(n_k l_k j_k),(n_l l_l j_l)}} = \overline{V}^{J',\mathrm{p}\Lambda}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l)} \\
&+ \sum_J \sum_{\substack{n_m l_m j_m \\ n_n l_n j_n}} \frac{2J+1}{2J'+1} \\
&\Bigg[\overline{V}^{T'_{12}=\frac{1}{2},T_{12}=\frac{1}{2},T=1,J',J',J}_{(n_i l_i j_i),(n_j l_j j_j),(n_m l_m j_m),(n_k l_k j_k),(n_l l_l j_l),(n_n l_n j_n)} \overline{\rho}^{\mathrm{p}}_{(n_n l_n j_n),(n_m l_m j_m)} \\
&+ \frac{1}{2} \overline{V}^{T'_{12}=\frac{1}{2},T_{12}=\frac{1}{2},T=1,J',J',J}_{(n_i l_i j_i),(n_j l_j j_j),(n_m l_m j_m),(n_k l_k j_k),(n_l l_l j_l),(n_n l_n j_n)} \overline{\rho}^{\mathrm{n}}_{(n_n l_n j_n),(n_m l_m j_m)} \\
&+ \frac{1}{2} \overline{V}^{T'_{12}=\frac{1}{2},T_{12}=\frac{1}{2},T=0,J',J',J}_{(n_i l_i j_i),(n_j l_j j_j),(n_m l_m j_m),(n_k l_k j_k),(n_l l_l j_l),(n_n l_n j_n)} \overline{\rho}^{\mathrm{n}}_{(n_n l_n j_n),(n_m l_m j_m)} \Bigg] .
\end{aligned}
\tag{B.28}
$$

$$
\begin{aligned}
&\overline{V}^{J',\mathrm{n}\Lambda,\mathrm{gen}}_{\substack{(n_i l_i j_i),(n_j l_j j_j)\\(n_k l_k j_k),(n_l l_l j_l)}} = \overline{V}^{J',\mathrm{n}\Lambda}_{(n_i l_i j_i),(n_j l_j j_j),(n_k l_k j_k),(n_l l_l j_l)} \\
&+ \sum_J \sum_{\substack{n_m l_m j_m\\ n_n l_n j_n}} \frac{2J+1}{2J'+1} \\
&\Big[\overline{V}^{T'_{12}=\frac{1}{2},T_{12}=\frac{1}{2},T=1,J',J',J}_{(n_i l_i j_i),(n_j l_j j_j),(n_m l_m j_m),(n_k l_k j_k),(n_l l_l j_l),(n_n l_n j_n)} \overline{\rho}^{\mathrm{n}}_{(n_n l_n j_n),(n_m l_m j_m)} \\
&+ \frac{1}{2} \overline{V}^{T'_{12}=\frac{1}{2},T_{12}=\frac{1}{2},T=1,J',J',J}_{(n_i l_i j_i),(n_j l_j j_j),(n_m l_m j_m),(n_k l_k j_k),(n_l l_l j_l),(n_n l_n j_n)} \overline{\rho}^{\mathrm{p}}_{(n_n l_n j_n),(n_m l_m j_m)} \\
&+ \frac{1}{2} \overline{V}^{T'_{12}=\frac{1}{2},T_{12}=\frac{1}{2},T=0,J',J',J}_{(n_i l_i j_i),(n_j l_j j_j),(n_m l_m j_m),(n_k l_k j_k),(n_l l_l j_l),(n_n l_n j_n)} \overline{\rho}^{\mathrm{p}}_{(n_n l_n j_n),(n_m l_m j_m)} \Big] .
\end{aligned}
\tag{B.29}
$$

The interaction matrix elements $\overline{V}$ in Eqs. (B.25) – (B.29), as well as the density matrices $\overline{\rho}$ are expressed in the self-consistent basis. I.e. they are transformed from the matrix elements in the HO basis by relations similar to Eq. (2.30). The three-body terms with total isospin $T = 0$ or 1 represent the elements of the ΛNN interaction and they are not implemented in the generalized matrix elements (B.25) – (B.29).

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