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Nucleon-nucleon interaction in the framework of effective theory

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Prohlášení

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V Praze dne:

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1 Introduction

Nuclear physics has witnessed in its long history many theoretical and phenomenological successes. Nevertheless, one of its basic ingredients, the nucleon-nucleon (NN) potential, is still not fully understood. While several successful models of the NN interaction exist, they are burdened by inherent purely phenomenological input and their connection to other hadronic processes and to the fundamental theory of the strong interaction is not sufficiently clear.

In the last decade, the studies of the hadronic interactions, in particular of the NN potential, gained a new momentum from developments of an effective field theory (EFT) approach. Very recently the NN potentials derived in this framework reached the level of agreement with the data, comparable to the best phenomenological ones. Still, some problems both in formulation and in actual implementation of this technique in derivation of nuclear forces persist. This thesis is our first step into this promising field.

This thesis can be divided in two parts: in the first part we – after short historical introduction – summarize our understanding of the current status of modeling the NN potentials (Chapter 2) and of the EFT approach (Chapter 3). Chapters 4 and 5 presents the technique (quantum mechanical formulation of the effective approach) which we learned in detail, including numerical implementation and tests.

Chapter 2 contains the historical introduction, after which more detailed description of the high-precision NN potentials follows.

In Chapter 3 explanation of the EFT begins. First, the basic principles are introduced and translated into the field-theoretical language. The description of the NN sector follows, it is explained why the NN interaction can be described in the EFT framework. Important part of this approach are regularization and counting rules, they are discussed in more detail. This Chapter is closed by the renormalization group equation approach applied to the NN sector.

Chapters 4 and 5 present the so-called effective theory of potential. This theory is a direct application of the effective approach to quantum mechanical framework. Chapter 4 introduces theoretical formulation of such approach, Chapter 5 presents numerical results.

Our conclusions are formulated in Chapter 6, some details of numerics are relegated to Appendices.

2 Historical overview

The story of the nuclear forces is a long and rich one, here we just remind the most important milestones and concepts, focusing on those related to the topic of this thesis. Much more comprehensive review can be found in [1,2].

The history of nuclear physics starts in 1911 by Rutherford, who first claimed the existence of the atomic nucleus as an explanation of the measured scattering behavior of alpha-particles on atoms. Further experiments followed, resulting in conclusion that nuclei are composed of nucleons: protons and neutrons. A natural question has arisen: What is the force holding them together?

In 1938 Yukawa put forward an idea that this force is mediated by an exchange of massive scalar particle. The non-zero mass of this particle ensures the finite range of the nuclear force, in agreement with experiment. It appeared that the natural candidate for the mediator of the nuclear force is a pion, experimentally observed in 1947. Taking into account that the pions are in fact pseudoscalars, one gets very promising contribution to the NN force already on the quasiclassical level from the exchange of a single pion (so-called one-pion-exchange (OPE) potential). However, it was known from the beginning that the pion-nucleon interaction (then considered to be fundamental) is strong, hence it was not clear how to systematically build up the full NN force from the corresponding Lagrangian.

In 1951 Taketani, Nakamura and Sasaki [3] introduced into this game an important organizing principle, based on the realization that the range of the particular force corresponding to an exchange of n pions is inversely proportional to the effective exchanged mass (roughly nm_π). Since the NN potential was already known to be strongly repulsive on very short distances, the multipion exchanges are suppressed and this justified the idea that only a finite number of contributions has to be considered, despite the strong coupling constant being much bigger than unity. Thus, they could divide the domain of the nuclear force into three regions:

i. a long-range ($r \geq 2\,\mathrm{fm}$), where the OPE potential dominates;
ii. an intermediate range, where exchanges of two and three pions prevail;
iii. the region of repulsive core ($r \leq 1\,\mathrm{fm}$) for which no simple mechanism was assumed and which was to be treated purely phenomenologically.

The next natural step was then to try to calculate the first important corrections to the OPE potential stemming from the two-pion exchange and test how well do they describe the intermediate range NN interaction. This attempt failed, since the authors required the underlying pion-nucleon field theory to be renormalizable, i.e., the pseudoscalar form of the pion-nucleon interaction had to be employed. These days we know that such interaction does not conform with the chiral symmetry. Nevertheless, the universal scheme which orders contributions to the total nuclear force according to their range survived and became an important part of the effective framework, described below.

Discovery of pion and nucleon resonances and a gradual realization that hadrons are composite particles opened a door for semi-phenomenological model building. The Yukawa's idea was extended to an exchange of other massive bosons. Various heavier vector and (pseudo)scalar mesons (mostly two- and three-pion resonances) were included and their contributions to different spin/isospin components of the NN potential were analyzed. The

short-range behavior was regularized by phenomenological form factors, their somewhat arbitrary functional form was usually constrained so that the potential could be easily treated in a coordinate representation. In late 1960's and in 1970's this approach resulted in successful description of the NN data by one-boson-exchange (OBE) potentials [4,5,6]. These potentials typically include just one meson with a given set of quantum numbers (parity, spin and isospin), with a mass between the pion mass and 1 GeV . This means that these exchanges most likely simulate all possible interactions in a given channel, rather than a single exchange of the observed meson/resonance. In some channels the corresponding contributions are rather well approximated by the exchange of single experimentally observed particle (e.g., ρ meson) with the appropriate mass and coupling constants, while for other channels (e.g., in that with an scalar-isoscalar exchange, represented by a fictitious σ-meson) an acceptably narrow resonance cannot be clearly identified.

The OBE picture was extended to two- and more-meson exchanges by the Paris and Bonn groups. The Paris group [7] developed potential in which they incorporated the dispersion relations to calculate the 2π-exchange. Results were complemented by the π and ω exchange. The short-range part of the NN interaction was treated as an energy-dependent repulsive square-shaped cut-off.

The Bonn group [6,8] analyzed multiple exchanges of pions and heavy mesons, together with contributions from nucleon resonances in intermediate states. The particular attention was payed to (partial) cancelations between certain (groups of) diagrams, the practical conclusion was partial justification of the OBE philosophy and families of non-relativistic and relativistic "Bonn" OBE potentials (in the coordinate and/or momentum representation) were developed, which fitted the data almost as well as the full Bonn potential.

In the meantime the fundamental theory of the strong interaction, the quantum chromodynamics (QCD), was established. In low and intermediate energy regions, i.e., in a realm of the hadronic physics, QCD is realized non-perturbatively and the direct QCD calculations of relevant hadronic observables are as yet not available. Several QCD-inspired approaches were applied also to the problem of the NN interaction, e.g., the constituent quark models [9], bag [10] or soliton models [11]. Such models typically succeed in explanation and semi-quantitative description of the short-range repulsive core, but longer-range features of the interaction, in particular the intermediate range attraction, have to be introduced via additional dynamical mechanism (e.g., meson-quark couplings), more or less by hands.

2.1 High-precision models of the NN interaction

Historically, the best models of 1980's were able to fit the data with the accuracy χ^2/datum ~ 2 or more. In 1993 the Nijmegen group [12] published the phase shift analysis for NN scattering up to 350 MeV with χ^2/datum ~ 1.0. This precise analysis caused a birth of a new generation of realistic NN models called high-precision potentials. In chronological order they are

1. Nijmegen Nijm-I, Nijm-II and Reid93 potentials [13];
2. Argonne V_{18} potential [14];
3. CD-Bonn potential [15].

All these potentials are inspired by the meson-exchange picture, in particular, all of them include the OPE part. They are also all charge-dependent, as required by the *pp* and *np* data.

The main differences among them are in meson exchanges explicitly included by respective groups and in parts of the force which are described completely phenomenologically. Also in the OBE components various approximations are made, in particular, to enable the Fourier transform of the corresponding Feynman amplitudes the non-relativistic reduction (resulting in the local approximation) is often employed.

The models Nijm-I and Nijm-II are based on the Nijmegen78 OBE potential [5]. Nijm-II uses the local approximation for all OBE amplitudes, while Nijm-I keeps some terms non-local. The Reid93 and V_{18} do not use meson exchanges for the intermediate and short-range parts of the potentials and describe them purely phenomenologically. The V_{18} potential employs the functions of the Wood-Saxon type, while Reid93 uses local Yukawa functions of multiples of the pion mass.

Unlike other models, the CD-Bonn employs the full nonlocal Feynman amplitudes for the OBE potentials. Apart from the pion, the physical vector mesons $\rho(769)$ and $\omega(783)$ and less important δ and η exchanges, two fictitious scalar-isoscalar σ-mesons are used.

All these models need 40-50 parameters to fit the data with the same accuracy of χ^2/datum ~ 1.0. However, to achieve such a good fit even the OBE models (CD-Bonn, NijmI and NijmII) had to adjust some coupling constants and/or cut-off parameters individually for various partial waves.

Since all these potentials are essentially phase-equivalent (and fitted to the deuteron binding energy), any difference between them can show up only when the NN system is either probed by external probes (typically photons or electrons) or embedded in larger nucleus. Ideally, calculating the 3N scattering, spectra of the lightest nuclei and their e.m. form factors and photo-/electro-disintegration would allow to distinguish between high-precision potentials. To accomplish this in practise some additional ingredients are necessary, both on the experimental and theoretical side. First of all, high precision data (in particular on sensitive polarization observables) are required; unfortunately not enough of them is available neither for the 3N scattering nor for the e.m. observables. As for the theory, the 3N forces and e.m. currents have to be constructed, fully consistent with particular model of the NN interaction. Although considerable progress in this direction was made in last two decades (see e.g. reviews [16]), this work is still in progress.

To sum it up, the current model description of the NN interaction, based on the meson-exchange picture with semi-phenomenological adjustments at the short-range can fit very successfully the NN data below pion production threshold. Extension of the underlying ideas to the 3N force and e.m. and weak nuclear currents is, in principle, also phenomenologically successful, but not yet as conclusive: neither of the approaches described above could be excluded and neither can claim a full success in describing all (somewhat scarce) available data. More important, it is rather difficult to deduce from these models some unambiguous information on related aspects of hadronic dynamics, e.g., on the importance of various physical (heavy) mesons and/or nucleon resonances, on their coupling constants, on whether it is adequate to include them as stable particles etc. It is also difficult to relate assumptions and the phenomenological input of these models to the underlying fundamental theory, the QCD. Even the important symmetries of the QCD – the approximate isospin symmetry and the approximate spontaneously-broken chiral symmetry – are reflected only purely phenomenologically by adopting the experimental values of the meson masses and coupling constants, considering the phenomenological mixing of heavier mesons and recognizing the

leading role of the pion as the lightest meson.

At the same time, it is firmly established these days that the chiral symmetry of the QCD is realized in the hadronic world in the Nambu-Goldstone mode, which explains the low mass of the pions (identified with the pseudo-Goldstone bosons of this broken symmetry), while the typical mass scale for all other (non-strange) hadrons is roughly 1 Ge V . The important dynamical role of the Nambu-Goldstone chiral symmetry was realized already in 1960's before the birth of the QCD and elaborated within the framework of the current algebra, leading to a number of predictions on processes involving "soft" pions. The current theoretical approach, which adopts and extends these dynamical ideas and allows to systematically explore corrections to soft-pion predictions, is the Chiral perturbation theory (ChPT) [17,18]. It is a field-theoretical formulation of the low-energy pion and pion-nucleon dynamics in the framework of an effective theory, which absorbs the short-range effects into contact counter-terms and defines the systematic perturbative expansion in terms of pion momenta. This approach is a basis of modern theoretical model-independent investigations [19] of the NN interaction (among many other processes in this energy domain) and will be reviewed in some detail in the next Chapter.

Before turning to the effective theory approach let us for completeness briefly mention a current state of direct calculations of the NN interaction using the lattice simulations [20]. The lattice simulations of these days cannot yet include the full dynamics of light quarks on the large enough lattice. Therefore, interactions of two heavy-light mesons are studied [21] and it is believed that this system will exhibit a similar dynamical features as the NN system (one can investigate, e.g., a role of the quark exchange). Clearly, significant improvement is still needed to extract the NN potential from the lattice.

3 Effective field theory

This Chapter introduces the effective field theory following reviews [22,23,24]. First, we explain what is meant by the effective theory and exemplify it on a case of a current source. The generalization into the field-theoretical language follows. Then we describe the NN sector and show why it is possible to describe it in terms of the effective theory. This concept is then defined more exactly, in terms of effective Lagrangians. Implementation of various regularization schemes is described and, finally, renormalization group equation is introduced as an universal tool to treat some problems that the effective field theory faces in the NN sector.

The effective theory is a general method to deal with a physical system in which different scales exist. What does it mean? Imagine a system with two separate scales, described by an underlying theory, which in turn can be roughly divided into two parts according to the scale it affects. Let us assume that we are interested in some process related to the first scale. Then we build up a description that includes explicitly only those parts of the theory which are connected to this scale, the remaining part connected with the second scale will be included indirectly through some representative terms. These terms are constrained only by predictions for the physical process at the first scale. There is no further restriction on their form, they are not required to converge to the true second-scale theory. This approach is helpful if the theory connected with the second scale is difficult to solve or not completely understood. Very rough analogy from mathematics is approximation of a well-behaved function by its Taylor expansion in a vicinity of some point: close to this point both the original function and its Taylor polynomial give the same results, so the polynomial can be used as a substitute for the full, more complicated, function.

Another, physical, example is from a classical electrodynamics. Let us consider some complicated current source $\mathbf{J}(\mathbf{r}, t)$ of the size d and try to describe only the part of the radiation it emits with wavelengths $\lambda \gg d$. For this purpose it is sufficient and suitable to mimic the source by a sum of point-like multipole currents ($E1$, $M1$, etc), usually just a few leading terms suffice to get an accurate approximation. This is exactly in a spirit of the effective theory. There are two scales in a problem, first one connected with size of the current source (d), the second one corresponding to the radiated wavelength (λ).

3.1 Field theoretical language

Explanation of the effective theory above is rough and heuristic. In the example of multipole expansion the effective approach serves as the tool for simplifying one particular problem. The language of the field theory enables us to use it in more universal way. Effective description of one particular process (more exactly, particular values of coefficients of the effective theory, analogous to values of multipole moments) can be used with the help of the Lagrangian formalism in all related processes.

Consider a field theory described by the Lagrangian $\mathcal{L}_{th}$ written in terms of some elementary fields Ψ. Suppose further that this theory describes well experiments in certain region of energy, the S-matrix elements can be calculated from the path integral

$$Z = \int \mathcal{D}\Psi e^{i \int \mathcal{L}_{th}(\Psi)} . \tag{3.1}$$

Let us now consider processes below some scale Λ. The states with momenta larger than Λ cannot be directly produced, so the elementary fields can be split into two parts according to whether their momenta are higher or lower than the scale Λ: $\Psi \to \Psi_h + \Psi_l$. Integrating over the faster component one gets

$$Z = \int \mathcal{D}\Psi_l \, e^{i \int \mathcal{L}_{eff}(\Psi_l)} \,, \tag{3.2}$$

in which $\mathcal{L}_{eff}$ stands for the effective Lagrangian that is in D spacetime dimensions defined as

$$\int \mathrm{d}^D \mathcal{L}_{eff}(\Psi_l) = -i \ln \int \mathcal{D}\Psi_h e^{i \int \mathcal{L}_{th}(\Psi_h, \Psi_l)} = \int \mathrm{d}^D x \sum_i g_i(\Lambda) \mathcal{O}_i(\Psi_l) \,, \tag{3.3}$$

where the operators $\mathcal{O}_i$ consist of the slowly-varying fields Ψ_l, because Ψ_h were integrated out.

Hence, the two main properties of $\mathcal{O}_i$ are: First, these operators are local, they contain only those fields that are placed at the same spacetime point. Second, they contain the arbitrary number of field derivatives. If the splitting is done carefully, the effective Lagrangian has the same symmetries as the original one (this is not true for the case of anomalous symmetries). If the original symmetry is spontaneously broken, the symmetry in the effective Lagrangian will be realized in a non-linear way.

The only information about fast dynamics in the effective Lagrangian is through the coefficients g_i. Their values depend on the cut-off, in accordance with the renormalization group equation, hence they are called "running coupling constants". The observables have to be independent on the cut-off.

The effective theory approach is particularly suitable for theories with at least one characteristic mass scale M. In such cases, the effective degrees of freedom for $\Lambda \leq M$ expressed by Ψ_l are significantly different from the ones from the original theory represented by Ψ.

The simplest example of the existence of the mass scale is when it corresponds to a mass of a physical particle. Production and decay of this particle do not concern the EFT because they involve large momenta. Effects of a virtual propagation of this particle are of short-range and are included only indirectly, i.e., into g_i's. If the underlying theory implies that the particle is not stable, then it does not appear at all at low energies and there is no need to include it. Another, a little bit more complicated common case is when the scale corresponds to some scalar field acquiring a non-vanishing vacuum expectation value and breaking a continuous global internal symmetry. Examples can be found in [25].

Let us now say a few words about the relation between the underlying and effective theories. If we know and can solve the underlying theory, we are (in principle) able to calculate effective theory up to any given precision. On the other hand, the Weinberg "theorem" [26] defines the role of the basic consistency principles and of the symmetries to be respected in the effective theory approach:

> In conciliating quantum mechanics and Lorentz invariance in a way consistent with unitarity, analyticity and cluster decomposition, we are led to quantum field theory; the most general Lagrangian with some assumed symmetries will then produce the most general S-matrix incorporating those general principles and symmetries, without any other physical content. This theorem has not been proven in general, however, there is not any known counter-example.

The EFT is a modern realization of the S-matrix approach, the only thing it can do is to relate observables. A field theory based on the Lagrangian formalism is the simplest way to do it, since we have given only some degrees of freedom and some symmetries. The theory can be regulated by implementing the cut-off's, that exclude some unwanted degrees of freedom. A renormalization proceeds as usual, however, we do not want to do away with infinities, i.e., to take the limit $\Lambda \to \infty$. The effective theory does not want to stay as the theory for all (especially high) energies. Nevertheless, the regularization in EFT has to remove details of the cut-off procedure from physical amplitudes. Only observables, which are extracted from on-shell amplitudes, are unambiguous in the EFT.

As stated above, not all possible terms are allowed, since we are restricted by the symmetries of the underlying theory. What symmetries do we have to take into account at low energies? We have to include the space-time symmetries, the system might also posses some internal symmetries. Although the symmetries restrict the theory, there is still an infinite number of terms. In order to keep the orderliness, the so-called power counting has to be introduced. It orders the contributions in a theory independent way. It is not obvious to have the situation in which the infinite number of coupling constants will be all small enough to make the coupling-constant expansion to converge. But there exists one possible small quantity which enable us to order the theory: a typical momentum Q.

3.2 NN sector

After this general discussion we move to the main topic of this review: effective theories in the NN sector. To employ the general procedure described above we have to proceed in steps as follows:

1. identify relevant degrees of freedom and symmetries,
2. formulate most general dynamics consistent with them,
3. define a systematic way of calculating the observables, usually as perturbative expansion in terms of some small parameter, complemented by the "power counting" scheme.

In the hadronic and nuclear sector, at least two scales can be recognized. One of them is connected with the characteristic QCD scale $\lambda_{QCD} \sim 1$ GeV. We observe that the masses of nucleon, of its excitations (e.g, Δ-isobar), and of most of the mesons are clustered around this scale. The second scale is roughly $\lambda_{nucl} \sim 100$ MeV. This scale is connected with the reciprocal size of light nuclei, with the Fermi momentum of an equilibrium nuclear matter and also with the pion mass. From these two scales, the third one can be built, $\lambda_{nucl}^2/\lambda_{QCD} \sim 10$ MeV, that is connected with nuclear binding energies.

These scales gives us a picture of nuclei of non-relativistic nucleons with typical momenta $Q \sim \lambda_{nucl}$ and energies $E \sim Q^2/\lambda_{QCD}$. This implies, that a nucleon in a nucleus cannot see below $\leq 1/Q$. In general, particles that live for short times and distances can be treated as pointlike. This is very significant: a large number of states, like massive hadrons, can be treated through local interactions. Only those particles which propagate over distances $\sim 1/Q$ has to be explicitly included into the theory as additional degrees of freedom. Further, the effective low-energy theory should posses the same symmetries as the principal theory, QCD. Unfortunately, there is still an infinite number of local interaction terms. If we want to keep our approach systematic, we have to find, how to order it. The parameters, characterizing the interaction, need not be small, thus the most useful possibility is to make an expansion in a small parameter Q.

First, let us examine, what happens in the simplest, low-energy regime, where $Q < \lambda_{nucl}$. In this case the nucleon momenta are so small, that all interactions can be described as local. However, a non-trivial complication occurs in the s-wave channels of the NN system. At low momenta, the scattering amplitude can be for both isospin channels (I=0 and I=1) described via the effective range expansion. The dimensional analysis of these parameters shows, that all but the first two of them (i.e., all except the scattering length and radius) scale with powers of λ_{nucl}^{-1}. However, the scattering lengths are large, hence there exists still another smaller mass scale $\aleph \sim 1/a$ [27]. Values of these new scales are $\aleph^{^3S_1} \sim 40$ Me V and $\aleph^{^1S_0} \sim 8$ Me V. For simplicity let us take an average $\aleph \sim 30$ Me V [22]. The existence of these scales is connected to appearance of shallow bound states with binding energies $\sim \aleph^2/\lambda_{QCD} \ll \lambda_{nucl}^2/\lambda_{QCD}$. In the singlet $^1S_0$ channel this shallow bound state is virtual, in the triplet $^3S_1$ channel it is, of course, the deuteron.

It might seem that the EFT in the two-nucleon sector in this energy range is more or less a simple extension of the work of Bethe and Peierls [28] from 1930's. However, the EFT treatment enables us to extend this approach to processes with more nucleons and to other hadronic processes.

If momenta are increased so that $Q \sim \lambda_{nuc}$, it is not advantageous anymore to treat pions as heavy particles. They enter into the game as the explicit degrees of freedom. The EFT in this energy region, the Chiral perturbation theory, has been successfully applied to none- and one-nucleon systems.

The use of the effective field theory as a technique to solve nuclear problems starts with Weinberg's seminal papers [29]. Weinberg proposed to use the power counting rules known from the chiral perturbation theory directly to n-nucleon potential rather than to the S-matrix. Only n-nucleon irreducible graphs should be included into the effective potential. The resulting effective potential can be inserted into the Lippmann-Schwinger equation or into the Schrödinger equation and unknown coupling constants has to be fitted to the experimental data.

The EFT for the NN scattering built in this way differs in a fundamental way from the EFT used for calculating processes like $\pi\pi$ scattering in the χPT. In both cases, terms in the Lagrangian are ordered in the same way. However, for the $\pi\pi$-like processes, the operator expansion in the effective Lagrangian maps into a power series in terms of k/M in the scattering amplitude. It can be proven, that the direct mapping from the effective Lagrangian to the S-matrix is systematic. For the NN problem the mapping from the effective Lagrangian to the effective potential applies, which is then iterated to all orders. In such a case, a question arises about the existence of a systematic power counting for the scattering amplitude (the T-matrix).

We will now write down the first terms of the most general effective Lagrangian relevant for derivation of the NN interaction and satisfying the chiral symmetry [30]

$$\begin{aligned}\mathcal{L} = {} & N^\dagger(iD_0 + \vec{D}^2/2M)N + \frac{f^2}{8}\mathrm{Tr}\partial_\mu\Sigma^\dagger\partial^\mu\Sigma + \frac{f^2 m_\pi^2}{2}\mathrm{Tr}(\Sigma + \Sigma^\dagger) \\ & - \frac{1}{2}C_S(N^\dagger N)^2 - \frac{1}{2}C_T(N^\dagger\vec{\sigma}N)^2 + \cdots\end{aligned} \tag{3.4}$$

where D_μ is a chiral covariant derivative and Σ is the exponential of isotriplet of pions

$$\Sigma = \exp\left(\frac{2i}{f}U\right), \quad U = \begin{pmatrix} \pi^0/\sqrt{2} & \pi^+ \\ \pi^- & -\pi^0/\sqrt{2} \end{pmatrix}, \tag{3.5}$$

and $f = 132$ MeV is the pion decay constant. The second line of the Lagrangian (3.4) contains only the purely-nucleonic contact terms. When the pion field is "frozen" the NN system is described with the help of only such terms. This is a situation we are considering in most of the discussion below, in particular the effective theory of the potential of the second part of the thesis is relevant just for this situation. For somewhat higher energies the pion degrees of freedom have to be included explicitly and the NN potential is built in the framework of the pion-nucleon chiral perturbation theory.

3.3 Power counting

The effective field theories rely on the existence of a separation scales, which distinguish between the low-energy physics and the short-distance physics. This makes it possible to systematically expand any observable in powers of Q/Λ_0, where the Q represents a generic low-energy scale and the Λ_0 is a typical scale of the underlying physics. The effective Hamiltonian or Lagrangian used for calculations of these observables can be arranged in a similar way. This expansion is useful, if the separation of scales is wide enough to enable the expansion to converge rapidly.

Power counting generally depends on a number of low-energy scales. Loop diagrams are of the same or higher order compared to the terms they are constructed of. Briefly said, power counting tells us, what to compute first. It enables us to perform calculations systematically: if we want to compute observables up to some order in the expansion coefficients, we need to involve into the effective theory only terms up to this considered order.

There are two different power counting schemes used in the NN sector, they are explained in more detailed below in the context of the renormalization group equation (RGE). Here we just briefly summarize their main features.

3.3.1 Weinberg power counting

Applying the Lagrangian described above to the NN scattering generates an infinite number of Feynman diagrams. However, Weinberg showed [29] that there exists a systematic expansion in terms of $(Q/\Lambda_\chi)^\nu$, where Q denotes a momentum or pion mass, $\Lambda_\chi \approx 1$ GeV is the chiral symmetry breaking scale and $\nu \geq 0$. This has become known as chiral perturbation theory (χPT); it was applied before in the zero- and one-nucleon sectors [16, 17]. We will see below, that the Weinberg power counting in the s-channel for NN system corresponds to an expansion around the trivial fixed point.

3.3.2 KSW power counting

When the interaction is strong a new low-energy scale can appear, which is generated by non-perturbative dynamics. An important example of this behavior is again the NN scattering with the unnaturally large scattering length in the s-wave channel. In such a case one has to sum up certain terms in the theory to all orders. This leads to a different counting scheme, often called Kaplan, Savage and Wise power counting (KSW) [31]. The KSW scheme corresponds to an expansion around the non-trivial fixed point of the renormalization group equation.

3.4 Regularization

We will restrict ourselves in the following only to the NN scattering in the ${}^1S_0$ channel for momenta $p \ll m_\pi$. Pions are above this scale and can be integrated out from the theory. The only remaining degrees of freedom are nucleon fields. Hence, the effective Lagrangian for the s-channel in the low-energy regime can be written as

$$\mathcal{L}_s = N^\dagger i\partial_t N - N^\dagger \frac{\nabla^2}{2M} N - \frac{1}{2} C_S (N^\dagger N)^2 - \frac{1}{2} C_2 (N^\dagger \nabla^2 N)(N^\dagger N) + HC + \cdots . \tag{3.6}$$

The Lagrangian above leads to an expansion of the potential in the form

$$V_{eff} = C_0 + C_2(k^2 + k'^2) + C_4(k^4 + k'^4) + C_4' k^2 k'^2 + \cdots . \tag{3.7}$$

The Georgi-Manohar naive dimensional analysis identifies the "naturalness" of the coefficients: $C_{2n} \sim m_\pi^{-2n-2}$. The potential (3.7), iterated via the Lippmann-Schwinger equation (LSE)

$$T(k', k; E) = V(k', k) + \int \frac{\mathrm{d}^3 q}{(2\pi)^3} V(k', q) \frac{1}{E - \frac{q^2}{M} + i\epsilon} T(q, k; E)\,, \tag{3.8}$$

has to be truncated at some finite order of k/m_π and k'/m_π, where according to the condition $k, k' \ll m_\pi$ the neglected terms are small. To get the two-nucleon quasi-bound state at the experimentally observed "unnaturally low" energy, one has to tune up the coefficients keeping them "natural".

When the potential is truncated at some finite order, infinities arise. The question is, whether all divergent terms can be regularized and renormalized in such a way, that momenta inside the loops are well below m_π.

We will remind now, how various regularization schemes work for the system defined above. The cut-off regularization [23, 24], the dimensional regularization with minimal subtraction [24] and, finally, the dimensional regularization with power-law divergence subtraction [31] will be considered.

3.4.1 Cut-off regularization

The idea of this scheme is to introduce the sharp momentum cut-off β, above which the new unknown physics occurs. All loops are then evaluated with momentum integrals up to β. The excluded part of physics has to be compensated for. These excluded modes are highly virtual, thus they can be replaced by a sequence of contact terms. For instance, if the cut-off is placed well below the mass Θ of some exchange quantum, the corresponding potential

$$V_\Theta(\mathbf{k}, \mathbf{k}') = \frac{g_\Theta^2}{(\mathbf{k} - \mathbf{k}')^2 + \Theta^2}, \tag{3.9}$$

can be for the momenta below the cut-off replaced by a contact interaction.

Let us place the cut-off below the pion mass $\beta < m_\pi$. The only possible degrees of freedom are then the nucleons with momenta below β. All higher-momenta states are integrated out. Corresponding effective potential includes form factors, that sharply cut the momentum

$$V_{eff}(p,p') = \left[C_0 + C_2(p^2+p'^2) + \cdots\right]\theta(\beta-p)\theta(\beta-p')\,, \tag{3.10}$$

where θ is the well-known step function. After the renormalization, resulting coefficients will depend on β.

The equation (3.10) is the infinite series, for real calculations it has to be truncated at some finite order. This causes the cut-off dependence of the scattering amplitude. This unwanted behavior can be canceled order-by-order by including higher order terms.

In what follows, (3.10) is put up to the second order, thus only two coefficients C_0 and C_2 survive. The LSE with such potential can be solved [23], one gets the inverse of the on-shell scattering amplitude:

$$\frac{1}{T(p)} = \frac{(C_2 I_3 - 1)^2}{C + C_2^2 I_5 + p^2 C_2(2 - C_2 I_3)} - \mathcal{I}\,, \tag{3.11}$$

where

$$I_n = -M\int \frac{\mathrm{d}^3 q}{(2\pi)^3} q^{n-3}\,, \tag{3.12}$$

and

$$\begin{aligned}\mathcal{I} &= -M\int \frac{\mathrm{d}^3 q}{(2\pi)^3}\frac{1}{p^2-q^2+i\epsilon}\\ &= I_1 - \frac{iMp}{4\pi} + Mp^2\mathcal{P}\int\frac{\mathrm{d}q}{2\pi^2}\frac{1}{p^2-q^2}\,.\end{aligned} \tag{3.13}$$

All integrals have upper bound equal to cut-off β, $p = \sqrt{ME}$ stands for the on-shell momentum. $\mathcal{P}$ means the principal value of the integral.

After renormalization up to $\mathcal{O}(p^4)$, one can identify the inverse of the scattering amplitude with the effective range expansion

$$\frac{1}{T(p)} = -\frac{M}{4\pi}\left(-\frac{1}{a} + \frac{1}{2}r_e p^2 + \cdots\right), \tag{3.14}$$

which yields the following set of equations for the coefficients C_0 and C_2:

$$\begin{aligned}\frac{M}{4\pi a} &= \frac{(C_2 I_3 - 1)^2}{C_0 + C_2^2 I_5} - I_1\,,\\ \frac{M r_e}{8\pi} &= \left(\frac{M}{4\pi a} + I_1\right)^2 \frac{C_2(2-C_2 I_3)}{(C_2 I_3 - 1)^2} + \frac{M}{2\pi^2\beta}\,.\end{aligned} \tag{3.15}$$

For fixed β these equations can be solved, the coefficients C_0 and C_2 will, of course, depend on β.

This dependence of the coefficients on the cut-off β can be determined [23] from the on-shell K-matrix:

$$K(p) = K(p,p;E) = \frac{N(p)}{N(p)A_0 M\beta + (1 + C_2 A_1 M\beta^3)^2}\,, \tag{3.16}$$

where the function in the numerator is

$$N(p) = C_0 + C_2^2 A_2 M\beta^5 + p^2 C_2(2 + C_2 A_1 M\beta^3)\,. \tag{3.17}$$

A_i are dimensionless integrals [23] that are positive and finite for $\beta \to \infty$.

In the limit $\beta \to \infty$ this term has to be finite, thus the coefficients have an asymptotic behavior:

$$C_0 \sim \frac{1}{M\beta}\,, \qquad C_2 \sim \frac{1}{M\beta^3}\,. \tag{3.18}$$

This scale dependence for C_0 is the same as that proposed by Weinberg [29].

This behavior does not arise for natural a, i.e., if $a \sim 1/m_\pi,\ \beta \ll m_\pi$. In such a case, the leading behavior of the coefficients is different

$$C_0 \sim \frac{1}{Mm_\pi}\,, \qquad C_2 \sim \frac{1}{Mm_\pi^2\beta}\,. \tag{3.19}$$

In this natural case all loop effects from C_2 are suppressed by at least $(\beta/m_\pi)^2$. Therefore, for $\beta < m_\pi$ the non-perturbative treatment is not necessary, the cut-off scheme with $\beta < m_\pi$ gives for natural coefficients a perturbative EFT, where loop diagrams are consistently suppressed. When the perturbative calculation are performed, regularization scheme becomes immaterial and renormalization can be done.

On the other hand, for unnatural case one gets from (3.18)

$$\frac{C_2}{C_0} \sim \frac{1}{\beta^2}\,. \tag{3.20}$$

The leading terms in (3.16) cancel out, thus the finite contribution comes from the subleading behavior. Keeping the analytical dependence of the coefficients on the cut-off, one gets

$$C_0 = \frac{\alpha_0}{M\beta} + \frac{\gamma_0}{\beta^2}\,, \qquad C_2 = \frac{\alpha_2}{M\beta^3} + \frac{\gamma_2}{\beta^4}\,, \tag{3.21}$$

where $\alpha_{0,2}$ and $\gamma_{0,2}$ are appropriate constants, which have to be tuned to physical values. The K-matrix gives for such parameterization the finite scattering length, however, it possesses no explicit energy or momentum dependence, thus it leads to zero effective range. This is related to the fact, that a Hermitean potential of the range R put into the Schrödinger equation results in an effective range constrained by [32]:

$$r_e \leq 2\left(R - \frac{R^2}{a} + \frac{R^3}{a^2}\right). \tag{3.22}$$

The limit $\beta \to \infty$ forces the effective range to go to zero. In the coordinate representation it corresponds to replacing the short-range potential by the expansion in the delta functions.

To produce the positive effective range, one can introduce dibaryon fields [33] or make the K-matrix energy-dependent. The energy dependence of the result can appear through the linear energy dependence of the sub-leading coefficients γ_i (a non-zero effective range can be obtained without explicit energy dependence of C's, but it turns out to be negative [34]). An energy dependent γ_0 leads to the energy appearing in the potential with a coefficient of the order $\sim 1/\beta^2$, while the leading momentum-dependence from the C_2 has coefficient $\sim 1/\beta^3$. This results in the absence of a systematic power counting.

It can be shown, that in the limit of infinite cut-off, the terms of different orders in the potential expansion contribute to the same order in the expansion of the K-matrix. Weinberg's power counting breaks down for such regularization scheme.

A naive possibility to treat the problem of the breakdown of the power counting is to keep the cut-off β finite. This leads to meaningful results only in natural systems, where $a \sim r_e$. Keeping the cut-off $\beta \ll 1/r_e$, one gets the proper scattering length without any fine tuning of the coefficients.

In unnatural systems, like for the NN scattering, where $a \gg r_e$, this does not hold and a systematic power counting cannot be obtained. However, this approach is still useful for analyzing the low-energy NN scattering in terms of only few parameters [35]. The second part of this work deals with numerical examples of such approach.

3.4.2 Dimensional regularization with minimal subtraction

Another traditional regularization scheme is the dimensional regularization with a minimal subtraction scheme ($\overline{\text{MS}}$). This regularization scheme is commonly adopted in field theories, since it respects chiral and gauge symmetries.

The equation for the scattering amplitude (3.11) holds for any regularization scheme, only the integrals I_i are defined in a different way. In DR with $\overline{\text{MS}}$ all power-law divergences vanish, hence $I_1 = I_2 = I_3 = 0$. The on-shell amplitude has a form

$$\frac{1}{T^{\overline{MS}}(p)} = \frac{1}{C_0^{\overline{MS}} + 2C_2^{\overline{MS}} p^2} + \frac{iMp}{4\pi}\,. \tag{3.23}$$

Renormalization leads to

$$C_0^{\overline{MS}} = \frac{4\pi a}{M}, \qquad C_2^{\overline{MS}} = \frac{\pi a^2 r_e}{M}\,. \tag{3.24}$$

As a consequence, K-matrix is given by the first Born approximation

$$K(k', k; E) = V(k', k; E)\,. \tag{3.25}$$

The problem for systems with shallow bound states is, that the on-shell K-matrix varies rapidly with the energy. Then, the low-momentum expansion of K (and of the potential) is valid only for $p < \sqrt{2/ar_e}$, which for unnaturally large a means a very narrow region. Although the DR with $\overline{\text{MS}}$ is systematic, it is not useful for the s-wave NN scattering because of its narrow range of validity.

3.4.3 Dimensional regularization with power-law divergence subtraction

Alternative renormalization scheme to the DR with $\overline{\text{MS}}$ is the DR with a power-law divergence subtraction (PDS), proposed by Kaplan, Savage and Wise [31]. In such a scheme, the loop integrals similar to (3.12) are redefined in D spacetime dimension

$$
\begin{aligned}
I_n(E) &= \frac{M}{(2\pi)^{D-1}} \left(\frac{\mu}{2}\right)^{4-D} \mathcal{P} \int \frac{q^{2n}}{p^2 - q^2} \mathrm{d}^{D-1} q \\
&= -\frac{M}{(2\sqrt{\pi})^{D-1}} \frac{(\mu/2)^{4-D}}{\Gamma\left(\frac{D-1}{2}\right)} \mathcal{P} \int_0^\infty \mathrm{d}x \frac{x^{(D+2n-3)/2}}{x - p^2} \\
&= -\frac{M p^{2n}}{(2\sqrt{\pi})^{D-1}} \frac{(\mu/2)^{4-D}}{\Gamma\left(\frac{D-1}{2}\right)} \Re\left[(-p^2)^{(D-3)/2}\right] \Gamma\left(\frac{D+2n-3}{2}\right) \Gamma\left(\frac{3-2n-D}{2}\right),
\end{aligned}
\tag{3.26}
$$

where μ is an arbitrary scale.

The last Γ function has a pole for $D = 3$ and any n. This is a signal of the logarithmic divergence in three dimensions or a linear one in four dimensions. Contrary to DR with $\overline{\mathrm{MS}}$, for which all $I_i = 0$, the PDS scheme keeps the first term non-vanishing, while the remaining ones are still zero. This additional terms has to be canceled by the extra counterterm with the same pole [23, 31].

The approach above gives the on-shell K-matrix

$$
K(p, p; E) = \left(\frac{1}{C_0 + 2p^2 C_2} + A_0 M \mu\right)^{-1}, \tag{3.27}
$$

yielding the scattering length

$$
\frac{1}{a} = \frac{4\pi}{M} \left(\frac{1}{C_0} + A_0 M \mu\right). \tag{3.28}
$$

From this it follows, that the PDS can yield an unnaturally large scattering length without requiring C_0 to be unnaturally small. As a result, the scale $\mu \gg 1/a$ can be chosen. The scale dependence of $C_{0,2}$ reads

$$
C_0 \sim \frac{1}{M\mu}, \qquad C_2 \sim \frac{r_e}{M\mu^2}. \tag{3.29}
$$

If one chooses the scale μ to be of the same order as the momenta of the processes considered $\mu \sim p$ and much less than the scale of the new physics $\mu \ll 1/r_e$, then C_0 has to be treated to all orders, while C_2 contribution is suppressed by pr_e. Higher terms in the potential are suppressed in the same way, i.e., by powers of pr_e. Thus the PDS gives the systematic power counting, which is different from the one proposed by Weinberg. The linearly divergent terms have the same coefficient in all loop integrals up to the power of p^2. Powers of energy, that multiply integrals, and powers of momentum inside integrals, contribute in the same way, thus there is no distinction between them in the potential, as is expected for a scheme with systematic power counting.

3.5 A renormalization group equation approach

3.5.1 Introduction

The renormalization group equation (Wilson's continuous or the "exact" renormalization group) enables us to study the scaling behavior of systems in a wide range of areas of physics. There is a number of papers [23,35] employing it for studies of the power counting in effective

field theories. In this section we try to summarize their results for the case of the s-wave NN scattering.

To formulate the RG equation, once first needs to renormalize and regularize the theory. In what follows the momentum cut-off regularization is used, i.e., $|\mathbf{k}| < \Lambda$. Its value has to be chosen so that it is above all low-energy scales and below the scales of the unknown high-energy physics. Else the exact value of Λ is arbitrary and the observables are required to be Λ-independent. All the physics above Λ has been "integrated out" and all its effects are included into the coupling constants of the effective field theory, which are Λ-dependent. Finally, we have to rescale all dimensioned quantities in units of Λ. Variation of coupling constants in the rescaled theory with Λ and their flow is then described by a first-order differential equation, the renormalization group equation.

When the system has a clear separation of scales, the rescaled coupling constants become independent of Λ in the limit of $\Lambda \to 0$. In such a limit, more and more physics is integrated out of the theory. In a RG language, the theory flows towards infra-red fixed points. If we are close to a fixed point, the deviations from it scale as a powers of cut-off. Since rescaling means that each low-energy scale present in the potential adds one power of Λ, we can define the power counting for our EFT.

The perturbations around a fixed point are classified according to sign of the powers ν of the cut-off. The first possibility with $\nu > 0$ is known as an "irrelevant" perturbation. Appropriate flow is towards the fixed point in the limit $\Lambda \to 0$. The second case is a perturbation with $\nu = 0$, called "marginal". This type is well-known from the conventional renormalizable field theories. It is responsible for logarithmic flow with Λ. The last third case is a "relevant" perturbation with $\nu < 0$. It causes a flow away from the fixed point in the limit $\Lambda \to 0$. We say, that the fixed point is stable, if all perturbations around a fixed point are irrelevant. On the other hand, a fixed point is unstable, if there are one or more relevant perturbations.

In what follows we describe two infra-red fixed points: the trivial one and the non-trivial one. The expansion around each of them leads to its own power counting rules. After the pure short-range potential, the presence of additional long-range part (assumed to be known) will be also discussed.

3.5.2 RG treatment for short-range forces

Here we review, how to build up a RG equation for the two-body scattering caused by short-range forces.

The starting point is the Lippmann-Schwinger equation (LSE) for the scattering of two particles of the mass M by a potential $V(k', k; p)$

$$T(k', k; p) = V(k', k; p) + \frac{M}{2\pi^2} \int \mathrm{d}q\, q^2 \frac{V(k', q; p) T(q, k; p)}{p^2 - q^2 + i\epsilon}\,, \tag{3.30}$$

where $T(k', k; p)$ is the off-shell T-matrix, k and k' are relative momenta and p is the on-shell momentum corresponding to the center of mass energy: $p \equiv \sqrt{ME}$. In this section, the energy dependence of T or V is indicated as a dependence on the momentum p (unlike in the previous section), since below the operators are decomposed into power series in terms of momenta.

The on-shell amplitude $T(p)$ is connected to the phase shift $\delta(p)$ by

$$T(p) = T(p,p;p) = -\frac{4\pi}{M}\frac{1}{p\cot\delta(p) - ip}\,. \tag{3.31}$$

For weak scattering at low energies, we can expand the on-shell amplitude above in powers of p. Alternatively, it is more convenient to expand the inverse of the amplitude in the usual form of the so-called effective range expansion

$$-\frac{4\pi}{M}\frac{1}{T(p)} + ip = p\cot\delta(p) = -\frac{1}{a} + \frac{1}{2}r_e p^2 + \cdots, \tag{3.32}$$

where a is the scattering length and r_e is the effective range. The scattering length satisfies the zero-energy limit condition

$$\delta(p) \to n\pi - ap, \text{ for } p \to 0\,. \tag{3.33}$$

As explained above, when the wavelength is larger than the range of the forces, we can replace these forces by the effective Lagrangian or Hamiltonian, that consists of the contact interaction terms only. In general, they will contain time derivatives (in the case of Lagrangian) or an energy-dependence in (the Hamiltonian framework). In the coordinate space, the effective potential will be expressed by δ-function and its derivatives, in the momentum space, it will be expressed by power series in squares of moments (i.e., in energy), namely

$$V(k',k;p) = c_{00} + c_{20}(k^2 + k'^2) + c_{02}p^2 + \cdots. \tag{3.34}$$

where c_{ij} are appropriate coupling constants to each order. How to organize the terms in the potential to enable the EFT to reveal its true power is answered by the RG.

Before we proceed further, we explain, why the potential (3.34) depends on energy. Let us consider the "obvious" form of the potential expansion only in terms of momenta

$$V(k,k') = C_0 + C_2(k^2 + k'^2) + \cdots, \tag{3.35}$$

These terms correspond to the contact interactions $(\bar\Psi\Psi)^2$ and $(\bar\Psi\Psi)(\bar\Psi\nabla^2\Psi)$ + HC. Evaluating one-loops diagrams with these interactions, one gets following integrals:

$$\int_0^\Lambda \frac{q^{2n+2}\mathrm{d}q}{E - \frac{q^2}{M} + i\epsilon}, \quad n \geq 2. \tag{3.36}$$

Going with the cut-off to infinity, one gets the divergent pieces multiplied with powers of energy, E^m, $m \leq n$.

These divergences can be canceled by allowing the coefficients in (3.35) to depend on energy: $C_0 \to C_{00} + C_{01}E$. † If such potential is used in the loops, where the internal nucleons

† Such terms are not usually taken into account in treatments of the NN scattering, where the energy dependence is usually eliminated by use of the so-called "equation of motion". This stands for the arrangement, where we swap between energy and momenta by a unitary transformation on the wave function or, in the field theoretical language, by a transformation of fields. These transformations carry the combination $p^2 - k^2$, which vanishes on-shell, thus we call it equation of motion.

are off-shell, the two terms $C_{01}E$ and $C_2(k^2+k'^2)$ are not equivalent. Consider the one-loop diagram as an example with insertion of C_{00} and one of the two terms. Energy-dependent piece generates only the $n=0$ divergent integral multiplied by E, where the momentum-dependent one gives the same term multiplied by k^2 but additional term, the $n=1$ divergent integral. As the cut-off goes to infinity, both bare couplings will be renormalized differently. For any finite number of loops, the renormalized result can be on-shell ($k^2=k'^2=ME$) rearranged to have the same form. But it holds only for finite number of loops.

The energy dependent terms in the potential also arise, whenever degrees of freedom are eliminated from a Schrödinger equation (e.g., in a coupled channel approach). The energy dependent contact terms are also required to renormalize the contributions from the two-pion exchange potential.

Instead of using T-matrix , we switch to the so-called reactance K-matrix. The off-shell K-matrix satisfies also the LSE, but with the Green's function obeying standing-wave boundary condition, i.e., the usual prescription with $i\epsilon$ in the integral over q is replaced by the principal value. Calculating T from K ensures the unitarity of the T-matrix for Hermitian V. The advantage of the reactance matrix is the fact, that it is real bellow all thresholds for particle production. The on-shell T-matrix and K-matrix are related via

$$\frac{1}{T(p)}=\frac{1}{K(p)}+\frac{iMp}{4\pi}\,, \tag{3.37}$$

from which it follows from (3.32)

$$\frac{1}{K(p)}=-\frac{M}{4\pi}p\cot\delta=-\frac{M}{4\pi}\left(-\frac{1}{a}+\frac{1}{2}r_e p^2+\cdots\right)\,. \tag{3.38}$$

The scattering through the contact potential leads to the divergent loop integrals in the LSE, it is therefore necessary to introduce some renormalization scheme (e.g., some of those described in the previous Chapter). We will adopt here the cut-off regularization scheme, which introduces a sharp momentum cut-off $q=\Lambda$ into the loop integrals.

The regularized LSE for the K-matrix reads

$$K(k',k;p)=V(k',k;p)+\frac{M}{2\pi^2}\mathcal{P}\int\limits_0^\Lambda \mathrm{d}q q^2\frac{V(k',q;p)K(q,k;p)}{p^2-q^2}\,, \tag{3.39}$$

where $\mathcal{P}$ again stands for the principal value of the integral.

Alternatively, the momentum cut-off can be introduced by the separable form factor, i.e., for the potential (3.35) is replaced by (the same can be made for the potential (3.34)):

$$V(k,k';p)=f(k/\Lambda)\left[C_0(E)+C_2(E)(k^2+k'^2)\right]f(k'/\Lambda)\,, \tag{3.40}$$

where the factor $f(k/\Lambda)$ satisfies $f(0)=1$ and falls rapidly for $k/\Lambda>1$. The coefficients C_0 and C_2 are now energy-dependent, for the reasons explained above. The regularized potential has a two-term separable form, the LSE can be solved analytically, which gives

$$\begin{aligned}K(k,k';p)=f(k/\Lambda)\,&\frac{1+\frac{C_2}{C_0}(k^2+k'^2)+\frac{C_2^2}{C_0}[I_2(E)-(k^2+k'^2)I_1(E)+k^2k'^2I_0(E)]}{\frac{1}{C_0}-I_0(E)-2\frac{C_2}{C_0}I_1(E)-\frac{C_2^2}{C_0}[I_2(E)I_0(E)-I_1(E)^2]}\times\\ &\times f(k'/\Lambda)\,,\end{aligned} \tag{3.41}$$

where $I_n(E)$ are integrals analogous to (3.36) with $i\epsilon \to \mathcal{P}\int$. Using the original sharp cut-off, one gets the same result (3.41) without the factor $f(k/\Lambda)f(k'/\Lambda)$. From these results and from (3.38) one can calculate a and r_e.

Let us go back to (3.39) and show, how the RGE is obtained. The potential V depends on the cut-off Λ and at the same time we demand that K be Λ-independent, since all scattering observables have to be independent on the cut-off. We can rewrite the LSE in a symbolic manner

$$K = V(\Lambda) + V(\Lambda)G_0(\Lambda)K\,. \tag{3.42}$$

Differentiating this equation in respect to Λ and keeping in mind that K is cut-off independent, one gets

$$\frac{\partial V}{\partial \Lambda}\left[1 + G_0(\Lambda)K\right] + V(\Lambda)\frac{\partial G_0}{\partial \Lambda}K = 0\,. \tag{3.43}$$

operating on both sides of this equality with $(1 + G_0K)^{-1}$ from the right results in

$$\frac{\partial V}{\partial \Lambda} = \frac{M}{2\pi^2}\,V(k', \Lambda, p, \Lambda)\frac{\Lambda^2}{\Lambda^2 - p^2}V(\Lambda, k, p, \Lambda). \tag{3.44}$$

The solution of this equation is constrained by boundary conditions sequent on the fact, that the effective potential should describe low-energy scattering data caused by the short-range interaction. The resulting function has to be analytic function of k^2 and k'^2 for small values of k and k'. In addition, if the energy lies below all particle production thresholds, we require also the analyticity in energy, i.e., in $E = p^2/M$. To sum it up, the resulting function has to be analytic in k^2, k'^2 and p^2.

To get the RG equation one more step has to be made: one has to rescale all variables in terms of Λ to get dimensionless momentum variables, e.g., $\hat{k} = k/\Lambda$. Defining the rescaled potential as

$$\hat{V}(\hat{k}', \hat{k}, \hat{p}, \Lambda) = \frac{M\Lambda}{2\pi^2}\,V(\Lambda\hat{k}', \Lambda\hat{k}, \Lambda\hat{p}, \Lambda)\,. \tag{3.45}$$

Substituting it into (3.44), one gets the RG equation for the effective potential $\hat{V}$:

$$\Lambda\frac{\partial \hat{V}}{\partial \Lambda} = \hat{k}'\frac{\partial \hat{V}}{\partial \hat{k}'} + \hat{k}\frac{\partial \hat{V}}{\partial \hat{k}} + \hat{p}\frac{\partial \hat{V}}{\partial \hat{p}} + \hat{V} + \hat{V}(\hat{k}', 1, \hat{p}, \Lambda)\frac{1}{1 - \hat{p}^2}\hat{V}(1, \hat{k}, \hat{p}, \Lambda). \tag{3.46}$$

The systematic expansion of the effective potential can be found, if we take the cut-off to zero. The potential goes to the infra-red fixed point of the RG. The solutions of (3.46), that do not depend on Λ, are called fixed points. We will now consider two of them (one trivial and one non-trivial), which are related to the power counting schemes described above.

Trivial fixed point

The trivial fixed point corresponds to the simplest solution of (3.46):

$$\hat{V}(\hat{k}', \hat{k}, \hat{p}, \Lambda) = 0. \tag{3.47}$$

The rescaled potential tends to the trivial fixed point as cut-off goes to zero. The appropriate K-matrix is zero, i.e., there is no scattering. The trivial fixed point as a solution of (3.46)

appears for systems with weak scattering at low energies, it is suitable for description of scale free processes.

Consider now perturbations around this trivial fixed point, that scales as definite power of the cut-off parameter. First, we linearize the RG (3.46) about the trivial fixed point. We are looking for the solution of the form

$$\hat{V}(\hat{k}', \hat{k}, \hat{p}, \Lambda) = C\Lambda^{\nu}\phi(\hat{k}', \hat{k}, \hat{p})\,, \tag{3.48}$$

where the ϕ's satisfy the eigenvalue equation

$$\hat{k}'\frac{\partial\phi}{\partial\hat{k}'} + \hat{k}\frac{\partial\phi}{\partial\hat{k}} + \hat{p}\frac{\partial\phi}{\partial\hat{p}} + \phi = \nu\phi\,. \tag{3.49}$$

The solution of (3.49) has to be well-behaved close to origin of momenta and energy, thus we have the resulting functions of the form

$$\phi(\hat{k}', \hat{k}, \hat{p}) = \hat{k}'^{2l}\hat{k}^{2m}\hat{p}^{2n}\,, \tag{3.50}$$

where ν are RG eigenvalues for which it holds $\nu = 2(l + m + n) + 1$, where l, m and n are non-negative integers. We immediately see that all eigenvalues are positive, hence the fixed point is stable. This means that if we start with any such potential, the flow takes it to this fixed point in the limit of $\Lambda \to 0$.

The perturbation of the potential in the above discussed manner can be cast into the form

$$\hat{V}(\hat{k}', \hat{k}, \hat{p}, \Lambda) = \sum_{l,m,n} C_{lmn}\Lambda^{\nu}\hat{k}'^{2l}\hat{k}^{2m}\hat{p}^{2n}\,. \tag{3.51}$$

The hermiticity of this potential can be assured by $C_{lmn} = C_{mln}$. The terms like $i(k^2 - - k'^2)$ vanish after the integration by parts in coordinate space, thus they need not be included. The corresponding unscaled form of the potential reads:

$$V(k', k, p, \Lambda) = \frac{2\pi^2}{M}\sum_{l,m,n} C_{lmn}k'^{2l}k^{2m}p^{2n}\,. \tag{3.52}$$

In a "natural" theory, the coupling coefficients are usually expressed in units of some scale connected with the underlying physics and it is also possible to choose the scale in the way that the coefficients are all of order unity. This scale also tells us, where the expansion breaks down.

It can be clearly recognized, that the contributions to the unscaled potential are organized in powers of energy and momenta. This is exactly what Weinberg proposed, with the order of each term being $d = \nu - 1$ [29].

The trivial fixed point and the behavior in its neighborhood can be applied for systems with weak scattering, i.e., those with the small scattering length. It was shown in [31,36] that this power counting gives a systematic results for dimensional regularization with minimal subtraction as well as for cut-off schemes. The reactance K-matrix is in such a case given by the first Born approximation for the unscaled form of the potential (3.52). In particular, the on-shell K-matrix reads:

$$K(p) = \frac{2\pi^2}{M}\left[C_{000} + (C_{100} + C_{010} + C_{001})p^2 + \cdots\right] . \tag{3.53}$$

where we can immediately imagine, how are the coefficients C_{lmn} connected to the observables.

Non-trivial fixed point

The next simplest fixed-point potential, we can take into account, is that depending only on energy (but not on momenta):

$$\hat{V} = \hat{V}_0(\hat{p}) , \tag{3.54}$$

which satisfies the RG equation

$$\hat{p}\frac{\mathrm{d}\hat{V}_0}{\mathrm{d}\hat{p}} + \hat{V}_0 + \frac{\hat{V}_0^2}{1-\hat{p}^2} = 0 . \tag{3.55}$$

General way, how to solve the momentum independent RGE is to divide it by $\hat{V}_0^2$. This yields a linear equation for $1/\hat{V}_0$

$$\hat{p}\frac{\mathrm{d}}{\mathrm{d}\hat{p}}\left(\frac{1}{\hat{V}_0}\right) - \frac{1}{\hat{V}_0} - \frac{1}{1-\hat{p}^2} = 0 . \tag{3.56}$$

The solution of this equation is the loop integral

$$\hat{I}(\hat{p}) = \mathcal{P}\int_0^1 \frac{\hat{q}^2\mathrm{d}\hat{q}}{\hat{p}^2-\hat{q}^2} = -1 + \frac{\hat{p}}{2}\ln\frac{1+\hat{p}}{1-\hat{p}} , \tag{3.57}$$

which is analytic in energy in the limit of $\hat{p}^2 \to 0$. Finally, we get the fixed point potential

$$\hat{V}_0(\hat{p}) = -\left[1 - \frac{\hat{p}}{2}\ln\frac{1+\hat{p}}{1-\hat{p}}\right]^{-1} . \tag{3.58}$$

The detailed form of the energy dependence of this potential depends on the particular choice of the cut-off. The fact that it goes to a constant in the limit of $\hat{p} \to 0$ is a generic feature. Unscaled potential corresponding to (3.58) is

$$V_0(p,\Lambda) = -\frac{2\pi^2}{M}\left[\Lambda - \frac{p}{2}\ln\frac{\Lambda+p}{\Lambda-p}\right]^{-1} . \tag{3.59}$$

At the origin $p = 0$ the potential (3.59) is proportional to Λ^{-1}, the general property which is true for any form of cut-off.

For the momentum-independent potentials such as V_0 the LSE has a simple form and the obtained K-matrix is infinite, i.e.,

$$\frac{1}{K(p)} = 0 . \tag{3.60}$$

In other words, we have a system with an infinite scattering length and a bound state with zero energy. Since the system has a bound energy equal exactly to zero, there is no scale

associated with it and this is why it is described by a fixed point of the RGE. This fixed point is especially interesting for the low-energy nuclear physics, where s-channel bound states or resonances near the threshold exists.

As for the trivial fixed point above, we can expand a potential in a small perturbation around the non-trivial fixed point $\hat{V}_0$, that will scale with definite powers of the scale Λ

$$\hat{V}(\hat{k}', \hat{k}, \hat{p}, \Lambda) = \hat{V}_0(\hat{p}) + C\Lambda^{\nu}\, \phi(\hat{k}', \hat{k}, \hat{p})\,. \tag{3.61}$$

Functions ϕ satisfy the linearized RGE:

$$\hat{k}'\frac{\partial\phi}{\partial\hat{k}'} + \hat{k}\frac{\partial\phi}{\partial\hat{k}} + \hat{p}\frac{\partial\phi}{\partial\hat{p}} + \phi + \frac{\hat{V}_0^2(\hat{p})}{1-\hat{p}^2}\Big[\phi(\hat{k}', 1, \hat{p}) + \phi(1, \hat{k}, \hat{p})\Big] = \nu\phi\,. \tag{3.62}$$

The eigenfunctions of this equation depend only on the energy and have a form

$$\phi(\hat{p}) = \hat{p}^{\nu+1}\hat{V}_0(\hat{p})^2\,, \tag{3.63}$$

where we restrict ourselves on analytic functions in p^2 for the limit of $p^2 \to 0$. This together with the fact that $\hat{V}_0^2$ is analytic gives the restriction on the power of $\hat{p}$ in (3.63). This leads to the condition for the RG eigenvalues $\nu = -1, 1, 3, \ldots$. We can immediately recognize the presence of one negative eigenvalue, that makes this fixed point unstable. This one negative eigenvalue implies, that only those potentials, which has the coefficient of the unstable eigenvector equal to zero, flow to the non-trivial fixed point. In other cases, when the "unstable" coefficient is non-zero, the potentials flow either to the trivial fixed point or to infinity. Potentials, which are near the surface of zero "unstable" coefficient, will initially flow to the neighborhood of the non-trivial fixed point, where they can be treated perturbatively, and then, when the Λ is small enough, are kicked away.

If we write down the expansion of the rescaled potential in powers of the rescaled energy

$$\hat{V} = a_0(\Lambda) + a_2(\Lambda)\hat{p}^2 + \cdots, \tag{3.64}$$

we can imagine the flow for $\Lambda \to 0$ in the following Fig. 3.1. There are two fixed points, first (the trivial one) at the origin, and second (the non-trivial one) at the point $(-1, -1)$. The bold lines represent the flow lines, that lie along the RG eigenvectors close to the fixed points. The dashed lines show general flows. It is apparent from this figure, that only potentials "lying" on the stable perturbation defined by vertical line $a_0 = -1$ flow to the non-trivial fixed point. The movement of the general case can be deduced from the figure.

The unscaled potential, that is close to the non-trivial fixed point, can be written down in terms of the perturbations defined above as

$$V(p, \Lambda) = V_0(p, \Lambda) + \frac{M}{2\pi^2}\sum_{i=0}^{\infty} C_{2i-1}p^{2i}V_0(p, \Lambda)^2\,. \tag{3.65}$$

When the LSE with this potential is solved, one gets the on-shell K-matrix

$$\frac{1}{K(p)} = -\frac{M}{2\pi^2}\sum_{i=0}^{\infty} C_{2i-1}p^{2i}\,. \tag{3.66}$$

Putting the first two coefficients equal to

$$C_{-1} = -\frac{\pi}{2a}, \qquad C_1 = \frac{\pi r_e}{4}, \tag{3.67}$$

we can immediately recognize the relation to the effective-range expansion (3.32). Therefore, this fixed point is often called the effective-range fixed point.

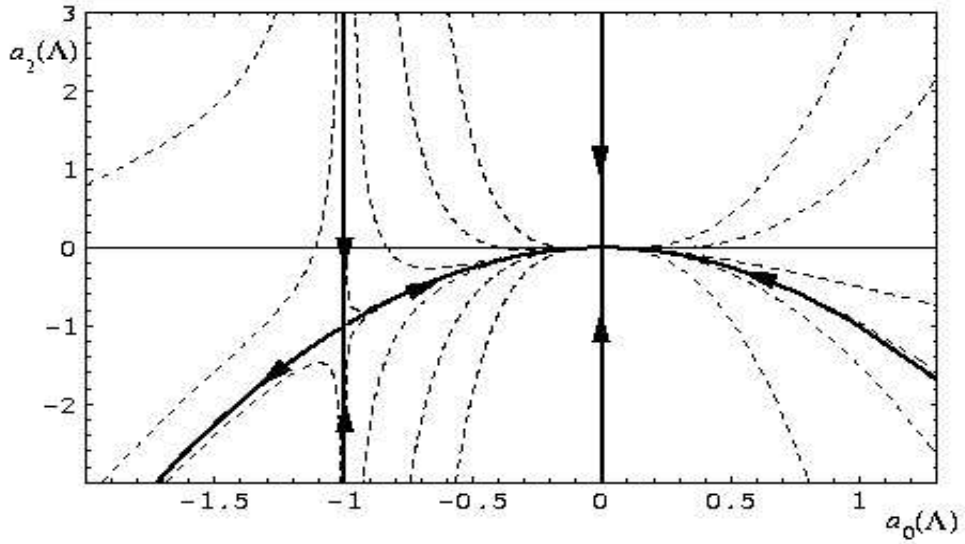

Fig. 3.1: The flow lines of the rescaled potential defined by (3.64) in the limit of $\Lambda \to 0$

Finally, let us mention the power-counting scheme for this fixed point. It was shown [31, 37] that one can assign orders to energy-dependent perturbations by $d = \nu - 1 = -2, 0, 2, \ldots$.

Lets say few more words about the particular case of s-wave NN scattering with unnaturally large scattering length a. From (3.67) it follows, that the coefficient C_{-1} is small and can be treated perturbatively. From the discussion above we know, that the effective potential will flow towards the non-trivial fixed point, when the cut-off Λ will be larger than $1/a$. To sum it up, the power counting defined by non-trivial fixed point is well suited for treating and organizing the terms in the NN potential, for momenta between $1/a$ and m_π.

3.5.3 Long-range forces

It is often possible to split the interaction of some particular problem into the known long-range part and the completely unknown short-range one. How does one includes such long-range interaction into RG formalism? One has to apply the cut-off to the basis of distorted waves of the known long-range potential. After applying the RG procedure, we can identify the possible fixed points of the short-range interaction and define the power counting rules for perturbation around it. If the non-trivial fixed points exist, we can directly relate the terms of the resulting EFT with the terms of the distorted wave effective-range expansion.

In general, if a long-range potential can be rescaled in a way, that leads to a scale independent form, we can apply the approach described above. Such a potential can be generated to all orders, then the basis of distorted waves can be obtained. This forms a starting point for further RG treatment. The fixed points correspond to the short-range

interaction. In combination with the long-range one, it leads in the limit where all low-energy scales tends to zero to a scale-independent behavior.

In order to implement the RG treatment for systems with long-range interaction, it is essential to identify all important low-energy scales of the system.

In the case of the Coulomb potential, the additional low energy scale is the Bohr radius (besides the momenta of the particles). Separation of scales between this and short-range effects is valid as long as this potential is relatively weak. If we express the Bohr radius in units of the cut-off, we ensure that Coulomb potential becomes scale independent when we rescale the theory. This implies, that it should be included as a part of any fixed point of the RGE. Thus, its effects should be implemented to all orders.

Next potential, which is of particular interest, is the OPE potential. It brings up another scale, the pion mass. It could be treated as a low-energy scales, if one is interested in momenta comparable to the pion mass. There are two ways how to do this. One was proposed on the KSW approach and treats these forces perturbatively, the second one relies on the principle introduced by Weinberg, in which these forces are iterated to all orders.

Now we will briefly formulate, what happens when we add the long-range part of the potential V_L, i.e., the full potential is given by

$$V = V_L + V_S \,, \tag{3.68}$$

where V_S represents the short-range physics which we want to parameterize.

In previous sections, we considered the RG equation, to get an information on the behavior of our system. Here we just write down the so-called distorted wave formalism, which is used to "project out" the short-range part from the long-range potential. Projected potential is treated in a similar way as that for pure short-range. The exact solution can be found in [23].

Distorted waves

First we write down the scattering amplitude for the long-range potential V_L alone

$$T_L = V_L + V_L G_L^+ T_L \,, \tag{3.69}$$

and the corresponding Green's function is

$$G_L^+ = G_0^+ + G_0^+ T_L G_0^+ \,. \tag{3.70}$$

Resuming V_L to all orders, the full T-matrix can be written as

$$T = T_L + (1 + T_L G_0^+)\tilde{T}_S(1 + G_0^+ T_L) \,, \tag{3.71}$$

where the distorted short-range scattering amplitude $\tilde{T}_S$ satisfies the LSE

$$\tilde{T}_S = V_S + V_S G_L^+ \tilde{T}_S \,. \tag{3.72}$$

The equation (3.71) is the starting point for the distorted-wave Born approximation, because the $\Omega = 1 + G_0^+ T_L$ is the Møller operator that transform plane waves into a distorted waves of V_L. The $\tilde{T}_S$ describes the scattering between the distorted waves of the short-range potential.

In terms of phase shifts the effect of the short-range potential is reflected by

$$\tilde{\delta}_S = \delta - \delta_L \,, \tag{3.73}$$

which is a difference between the full phase shift δ and δ_L that is due to the long-range potential alone.

The matrix elements of the on-shell $\tilde{T}_S$ between distorted waves can be written as

$$< \psi_L^-(p)|\tilde{T}_S(p)|\psi_L^+(p) >= -\frac{4\pi}{M} e^{2i\delta_L(p)} \frac{e^{2i\tilde{\delta}_S(p)} - 1}{2ip} \,, \tag{3.74}$$

where $\psi_L^+(p,r)$ is the outgoing distorted wave with the energy $E = p^2/M$ and $\psi^-{}_L$ is the incoming wave (again, only s-wave scattering is considered). From (3.74) so-called distorted-wave effective-range expansion is defined:

$$\begin{aligned} & e^{2i\delta_L(p)}|\psi_L^+(p,0)|^2 \left(\frac{1}{< \psi_L^-|\tilde{T}_S(p)|\psi_L^+ >} - ip \right) + \mathcal{M}_L(p) \\ & \quad = |\psi_L^+(p,0)|^2 p[\cot\tilde{\delta}_S(p) - i] + \mathcal{M}_L(p) = -\frac{1}{\tilde{a}} + \frac{1}{2}\tilde{r}_e p^2 + \cdots \end{aligned} \tag{3.75}$$

where $\mathcal{M}_L(p)$ is the logarithmic derivative at the origin of the Jost function to the Schrödinger equation with the long-range potential alone. An effective field theory for the strong interaction in pp scattering can be constructed [38] by taking the Coulomb potential as the long-range distorting potential. This expansion has been also used to remove the one-pion exchange effects from NN scattering [39].

This generalized effective range formalism is useful because all rapid dependence on the energy is removed in $\mathcal{M}_L(p)$ and $e^{2i\delta_L(p)}|\psi_L^+(p,0)|^2$. This leaves the scattering amplitude which can be expanded in powers of energy, where coefficients are scaled by the underlying short-distance physics.

4 The effective theory of potential I

Formalism

4.1 Introduction

In the first part of this thesis we were dealing mostly with effective approach within the framework of the filed theory. The field-theoretical treatment has a great advantage in its universality: once the coefficients of the Lagrangian are determined from some particular process(es), this knowledge is automatically available for any other process which contains the same dynamics.

However, the similar approach can be used in an entirely quantum-mechanical framework. This is described in detail in a pedagogical way in lectures by Lepage [40]. In this part of the thesis we will summarize this quantum-mechanical variation of the effective approach and test it numerically on several potentials.

We will deal with a system, whose dynamics can be described via the non-relativistic Schrödinger equation: one non-relativistic particle in a field of some potential. The considered method allows to systematically approximate the dynamics of such a system, we will dub it "the effective theory of potentials".

The effective theory of potentials is based on the paradigm of the renormalization theory: the low-energy (infrared) behavior of a theory is independent of the details of the short-distance (ultraviolet) dynamics. In other words, there are infinitely many theories that have the same low-energy behavior, i.e., they are identical at long distances, but they can essentially differ at short distances. We will employ this in a following way: we replace the true ultraviolet behavior of the potential with something simpler, keeping the same infrared observables. This simpler part will consist of the polynomial expansion in energy (or the square of momentum). Alternatively, in the coordinate representation, it will be described by the polynomial in derivatives of the contact interaction: the δ-function.

Our "simpler" approximation will also have an important characteristic of approaches from previous parts of the thesis: it will be made in a systematic way. That is, we are able (in principle) to increase the accuracy of our approximation up to any given order in a universal way.

4.2 Construction

In this part, we will remind [40], how to construct the systematic approximation described above. Let us consider a dynamical system described by the Hamiltonian

$$\mathcal{H} = \frac{\mathbf{p}^2}{2m} + V(\mathbf{r})\,, \tag{4.1}$$

where m is the (reduced) mass and the potential V can be divided into two parts

$$V = V_L + V_S\,, \tag{4.2}$$

where V_S is an unknown short-range potential, we want to simulate, and V_L is long-range potential, assumed to be known.

To test the algorithm of the effective theory of the potential, as it was done in [40], we will produce the "pseudo-experimental" data from the Schrödinger equation. Data are represented by bound state energies and by phase shifts. In the next step the effective theory is used. We will implement it through the cut-off a in the coordinate space. This cut-off has to be large enough to separate off the region of the unknown physics, represented by the potential V_S. The real physics behind the cut-off will be replaced by the contact interaction terms. It means, that the unknown physics is not omitted, but rather it is integrated out and replaced by the simplified dynamics.

In what follows we are using the familiar Coulomb-like potential for the long-range interaction, i.e., $V_L(r) = -\alpha/r$, and a series of various potentials for the unknown short-range part. We will restrict ourselves only to the scattering and bound states in the s-wave.

4.3 Naive approximation

The crudest approximation is to replace the short-range part by a δ-function. The approximate Hamiltonian is then of the form

$$\mathcal{H}_{app} = \frac{\mathbf{p}^2}{2m} - \frac{\alpha}{r} + c\delta^3(\mathbf{r}), \tag{4.3}$$

where c is a parameter representing the ultraviolet behavior. Using the perturbation theory in its first-order, we get for the energy levels

$$E_n^{app} = E_n^{Coul} + c\left|\psi_n^{Coul}(0)\right| = -\frac{1}{2n^2} + c\frac{\delta_{l,0}}{\sqrt{\pi}n^3}\,. \tag{4.4}$$

Parameter c has to be determined from the available data. As argued in detail in [40] it is advisable to match it on the binding energy of the state with large radial quantum number. This yields a relatively good approximation also for other binding energies and for phase shifts at low energies. However, opposite to the strategy outlined in the introduction, it is not possible to systematically improve this approximation. Indeed, by straightforward application of the second-order of the perturbation theory for the energy levels, one gets the corrections

$$\sum_{m\neq n} \frac{< n|c\delta^3(\mathbf{r})|m >< m|c\delta^3(\mathbf{r})|n >}{E_n - E_m}. \tag{4.5}$$

This expression gives an infinite correction to the binding energy, also the sum over scattering eigenstates diverges in the limit of the scattering momentum $\mathbf{p} \to \infty$. The delta function is too singular for usage beyond the first-order perturbation theory.

Approximation of V_S by the singular δ-function can be understood by considering the Fourier transformation of the short-range potential. Since V_S has short-range, its Fourier transform $v_S^F(q^2)$ depends weakly on the momentum transfer q. Thus we can make its Taylor expansion and approximate the transform only with a few first terms:

$$v_S^F(q^2) = v_S(0) + q^2 v_S'(0) + \cdots. \tag{4.6}$$

This replaces the Fourier transform of the potential by a few coefficients $v_S(0)$, $v'_S(0)$,.... The first term corresponds to the δ-function in the coordinate representation. Next term contains q^2, which corresponds to ∇^2 in coordinate space acting on the δ-function, and so on. Thus, up to the second order the potential is approximated by

$$V_S(\mathbf{r}) \sim c\delta^3(\mathbf{r}) + d\nabla^2\delta^3(\mathbf{r}). \tag{4.7}$$

This potential yields an infinite first-order correction to E_n^{Coul}, the derivative of the δ-function is too singular.

4.4 Effective theory

The infinities arising in the previous section are of the same kind as infinities in the relativistic QFT. They indicate, that the low-energy physics "feels" the short-range physics, however, not its details.

Modifying the results above, one can construct the effective theory, that is able to model the data to any precision. It is done in three steps

1. The long-range part of the underlying potential has to be known and is thus built into the effective theory explicitly;
2. The ultraviolet cut-off has to be introduced. It has two effects. First, it excludes the high-momentum states sensitive to the short-range physics. We can retain only states, we know well. Second, it regularizes all interaction at the origin, therefore it avoids all infinities arising in the naive approximation.
3. Local correction terms has to be added into the potential to mimic the effects of the short-range physics excluded by the cut-off. Each correction term consists of a theory-specific constant multiplied by a theory-independent local operator. The correction terms systematically remove the dependence on the cut-off, hence the physical observables will become (approximately) cut-off independent. Locality assures the finite number of terms needed to reach the given precision.

To avoid the infinities one has to regularize the δ-function (second part of the point two of the above itemization). This can be achieved by smearing the δ-function over the volume of the radius, which is approximately of the same size as the cut-off a. A possible smearing, used in most of the applications below, reads

$$\delta_a^3(\mathbf{r}) \equiv \frac{1}{(2\pi)^{3/2}a^3} e^{-r^2/2a^2}\,, \tag{4.8}$$

where we have to repeat, that the particular form of the smearing is not relevant.

Just one note has to be added now. The smearing of the δ-function corresponds to the cut-off regularization scheme used in the QFT, thus we are in fact renormalizing the Schrödinger equation or using the renormalization technique in the quantum mechanical problem.

Using the smearing defined by (4.8), we get an explicit form of the effective potential

$$\begin{aligned} V_{eff}(\mathbf{r}) = &-\frac{\alpha}{r} + ca^2\delta_a^3(\mathbf{r}) + d_1a^4\nabla^2\delta_a^3(\mathbf{r}) + d_2\vec{\nabla}\cdot\delta_a^3(\mathbf{r})\vec{\nabla} + \cdots \\ &+ ga^{n+2}\nabla^n\delta_a^3(\mathbf{r}) + \cdots \end{aligned}\,, \tag{4.9}$$

where the coefficients c, d_1, $d_2 \ldots$ represent dimensionless coupling constants, to be fitted from the low-energy data. As was mentioned above, the low-energy particle cannot distinguish between the contact terms and the real short-distance potential.

The effective potential also has to posses the symmetry as the original theory. Thus, if the data are not rotationally invariant, one has to include terms like $a^3\mathbf{g}\cdot\vec{\nabla}\delta_a^3(\mathbf{r})$. The couplings would contain vectors, tensors, etc., that will reflect the asymmetry of the original theory.

The original scattering amplitude can be written as

$$< f|T(E)|i >=< f|V|i > + \sum_n \frac{< f|V|n >< n|V|i >}{E - E_n} + \cdots. \tag{4.10}$$

The sum over intermediate states in second and higher orders contains states with arbitrarily high momentum. Thus, the Taylor expansion of the potential in momentum space would not converge in matrix elements involving such states. Now, we will replace real potential V by the effective potential V_{eff}, where we have introduced ultraviolet cut-off that actually restricts the sums to states with momenta that are not higher than $1/a$. Higher momenta contributions are completely absent in the effective theory.

If we assume that the final and initial states have low energy, the high-momentum intermediate states are highly virtual. The uncertainty principle tells us that these states cannot propagate for long time over a large space. Such states contribute to the amplitude through the same δ-terms, as we already have in the effective potential. Thus, the high-energy states are explicitly excluded from the theory by the cut-off, they are implicitly included through the contact terms. This means, that the coupling constants in our theory depend non-linearly on the true potential.

In the construction of the effective potential the cut-off a was introduced. The real theory does not depend on this parameter, but the effective theory is only approximately cut-off independent. This residual dependence enters in powers of (qa), where q is a characteristic momentum of the process considered, e.g., the momentum transfer in the scattering. That the effective theory is systematic also means, that the contact terms remove the a-dependence order-by-order in a. The leading term of $V_{eff} \sim \mathcal{O}(a^2)$ removes errors of order $(qa)^2$.

The coupling constants also depend on a, they have to be adjusted to compensate quantum fluctuations excluded by the cut-off. As in a field theory, they are called "running coupling constants". If the short-range potential is weak, the coupling to high-energy momentum states is weak and coupling constants vary slowly with cut-off. However, the couplings are considerably a-dependent, if the short-range potential is strong. This dependence can be non-linear and there could occur an exchange of the significance among the contact terms, i.e., a^4-term can play more significant role than a^2-operator.

What happens if we take too large cut-off? If its value is larger than the range of the long-range potential (or if there is no such potential), the coupling constants of the contact operators become a-independent. On the other hand, for too small cut-off, one gets the non-linear behavior described in the previous paragraph. From the motivation of the effective theory it is clear, that it makes no sense to take $a < r_S$. For such cut-off the high-energy states are explicitly included and the theory is through these states sensitive to short-distance dynamics. But it feels almost certainly wrong short-distance structure. In general, one can say, that to take $a < r_S$ does not improve the theory, rather it can cause essential problems

like non-stability or untuneability.

To close a brief description of the effective theory: The effective theory gives us a relatively simple, universal parametrization modeling the short-structure effects. The form of the contact terms does not depend on theory, only the numerical values of coupling constants are specified by the problem considered.

5 The effective theory of potential II

Numerical calculation

The effective theory of potential was defined and discussed in the previous Chapter. Now numerical results follow. The idea was to test numerically this quantum mechanical technique, following the example considered by Lepage [40]. Comparing to [40] we consider not just one example, but the whole set of "unknown" short-range potentials, differing by their behavior near the origin, by their smoothness, strength and by number of bound states. We also tried to apply the technique to the Schrödinger equation with only the short-range potential. Our hope is that at least some experience, e.g. with a multi-parameter fit of coupling constants, will be useful in the future. The effective theory of potential is just a kind of quantum mechanical motivation for further step to our future studies of similar problem in the quantum field theory framework, where the numerical tools and the gained experience could also be utilized.

The effective theory of the potential is used here up to the second order, i.e., up to the coefficient d_1 in (4.9), higher order terms are omitted (coupling coefficients are equal to zero). All employed potentials are spherically symmetric (and we restrict ourselves to the $s-$wave), thus (4.9) up to second order gets the form

$$V_{eff}(r;c,d_1) = \frac{-\alpha}{r} + ca^2\frac{e^{-r^2/2a^2}}{(2\pi)^{3/2}a^3} + d_1a^4\left(\frac{r^2}{a^4} - \frac{3}{a^2}\right)\frac{e^{-r^2/2a^2}}{(2\pi)^{3/2}a^3}\,, \tag{5.1}$$

where the smearing of the δ-function is chosen as given by (4.8). We also check an alternative form of smearing. The convention of $\hbar = c = 1$ is used. The mass is set to $m = 1$ and the strength of the Coulomb-like interaction is $\alpha = 1$. The value of the cut-off parameter a is set to $a = 1$ fm to start with, we discuss a sensitivity of the results in respect to its variation.

To simplify the notation we include the constant numerical factors at each order into the coefficients to get (for $a = 1$ fm) a simple form of (5.1):

$$V_{eff}(r;c,d_1) = \frac{-\alpha}{r} + ce^{-r^2/2a^2} + d_1(r^2-3)e^{-r^2/2a^2}\,. \tag{5.2}$$

Numerical tests of validity of the effective theory of potential are divided into three parts. In the first one, the effective theory is constructed for the chosen set of potential. In the second one we briefly illustrate effects of different smearing and of the cut-off variation. Finally, we apply the effective approach to purely short-range potential.

5.1 Construction of effective potential

We will built up the effective theory of the potential in the first and second order. We have to tune up the coefficients of the effective potential to some experimental data. We will produce our own pseudodata (binding energies and phase shifts) employing a set of model potentials.

Phase shift can be determined in two ways: 1) one can either solve non-linear first-order differential equation of the Variable phase method (Appendix A), 2) or the direct numerical

solution of the second order differential Schrödinger equation is matched to the asymptotical behavior of wave functions.

We tried both methods and then gave preference to the first one. One additional complication is the Coulomb-like behavior of our long-range potential. We could define "the strong phase shifts" in a usual way by subtracting the Coulomb "phase shift", but for our purposes it is more convenient to follow [40] and define the "pseudo phase shift" by cutting off the long-range potential at some very distant point (taken at 50 fm), way outside the region of the short-range interaction. Phase shifts defined in this way do, of course, contain all necessary information from the short-range potential [41].

Binding energies are computed by matching the numerical outward solution of the Schrödinger equation starting from the origin with the inward one, which is initialized by the asymptotical behavior of the wave function.

Numerical aspects of employing both methods are discussed in the Appendices. Let us now summarize the whole procedure of construction the effective theory in a series of steps:

1. Generation of pseudoexperimental data.
   a. Choose model potential
   b. Compute phase shifts and binding energies
2. Tuning the effective theory
   a. First order: tune c
   b. Compute phase shifts and binding energies
   c. Second order: tune c and d_1
   d. Compute phase shifts and binding energies
3. Comparing results
   a. Plot graphs, compile tables

1a. Model potentials

The short-range potential we used are listed below, each potential is labeled by capital letters. A short motivations for using this or that particular form are given.

First is the set of repulsive square-well potentials, for which the phase shifts and binding energies are known analytical results. They also represent non-continuous potentials, which is a good check of our numerics (although the phase shifts and energies are known, we also recalculated them numerically).

$$V_S(r) = \begin{cases} -0.2\ \mathrm{MeV}, & r \le 1\ \mathrm{fm} \\ 0\ \mathrm{MeV}, & r > 1\ \mathrm{fm} \end{cases} \qquad (A)$$

$$V_S(r) = \begin{cases} -0.4\ \mathrm{MeV}, & r \le 1\ \mathrm{fm} \\ 0\ \mathrm{MeV}, & r > 1\ \mathrm{fm} \end{cases} \qquad (B)$$

$$V_S(r) = \begin{cases} -2\ \mathrm{MeV}, & r \le 1\ \mathrm{fm} \\ 0\ \mathrm{MeV}, & r > 1\ \mathrm{fm} \end{cases} \qquad (C)$$

$$V_S(r) = \begin{cases} -4\ \mathrm{MeV}, & r \le 1\ \mathrm{fm} \\ 0\ \mathrm{MeV}, & r > 1\ \mathrm{fm} \end{cases} \qquad (D)$$

The following two potentials are "motivated" by the Yukawa one.

$$V_S(r) = \frac{e^{-20\,r}}{r} - \frac{e^{-0.4\,r}}{r}\,, \qquad (E)$$

$$V_S(r) = \frac{e^{-15\,r}}{r} - \frac{e^{-r}}{r}\,. \qquad (F)$$

Next two potentials are chosen because their are smooth, fall off rapidly and their shape is similar the smeared delta function:

$$V_S(r) = -7e^{-5\,r^4}\,, \qquad (G)$$

$$V_S(r) = -4e^{-5\,r^4}\,. \qquad (H)$$

This potential is singular at the origin and falls off relatively slowly for large distances:

$$V_S(r) = -\frac{0.1}{r^2}\,. \qquad (I)$$

Last two potentials represents repulsive short-range potentials:

$$V_S(r) = +e^{-15\,x^4}\,, \qquad (J)$$

$$V_S(r) = +2e^{-15\,x^4}\,. \qquad (K)$$

1b. Phase shifts and binding energies

Next step is calculation of phase shifts and binding energies for each of these potentials. High precision of the pseudodata is required for tuning of the effective potential, all computations have to be performed with double precision numerics. More detailed description of numerical computations is given in Appendix C. We have produced the binding energies with the accuracy of 10^{-9} and phase shift of 10^{-7}.

2a. Tuning the effective potential at the first order

As explained in the theoretical introduction (previous Chapter), the constants of the effective potential are fitted to the experimental data. At the first order there is just one constant c, all remaining ones are put to zero. We have tuned the c by matching the phase shift for the energy 10^{-9}, as described in the third part of Appendix C.

2b. Phase shifts and binding energies for 1st order V_{eff}

Calculated as for the full "unknown" potential, as described in point 1*b*. above. This applies also for the point 2*d*., i.e., for calculation with the second order V_{eff}.

2c. Tuning the effective potential at the second order

Tuning the coefficients of the second order effective potential represents finding a solution of a two-dimensional minimalization problem. The coefficients c and d_1 were matched on the phase shift for two energies, 10^{-9} and 10^{-5}. Since we do not know an analytical dependence of the minimalized function on these parameters, this has to be done numerically and the numerical implementation is non-trivial. All employed procedures are yielding results dependent on the initial guess. Detailed discussion follows is given in Appendix C.

5.2 Results

In the this section we present our numerical results. First, we will summarize the coefficients of the first- and second-order effective theory for all potentials employed. The binding energies and phase shifts for one of them are listed. This is followed by a picture with the shapes of the model potential together with the first- and second-order effective ones. Then we illustrate the problem of the minimalization procedure. The variation of the cut-off and of the form of the smeared δ-function is discussed. Finally, the results of the effective theory for purely short-range potential are given.

Potential	c (1oET)	c (2oET)	d_1
A	-0.19553534	-0.11703258	0.03895575
B	-0.38848651	-0.28305963	0.05584791
C	-1.3319855	-1.21876579	0.08458999
D	-1.8430867	-1.72016913	0.08270194
E	-2.4826996	-2.35926803	0.09264444
F	-1.4384087	-1.25577909	0.14187477
G	-2.1048997	-2.00463607	0.07831975
H	-1.626638	-1.59442330	0.02475360
I	-1.2531224	-1.12278586	0.09599282
J	0.21441655	0.19329079	-0.00984438
K	0.36534588	0.31938749	-0.02161320

Tab. 5.1: The coefficients for the first- and second-order effective theory for the set of potentials A – K.

We can see that the coefficient c vary only slightly from the first to the second order each order, the coefficient d_1 is much smaller.

Since results of the effective theory appear qualitative similar for each of the potentials A – K used, i.e., the effective theory of potential works equally well for all considered cases, we will focus in more detailed discussion below on one particular case: the potential H.

We will consequently show in two graphs and tables below, how the effective theory approximates the binding energies and phase shifts at each order. Recall that we have used for matching of the first-order effective potential the phase shift at the energy 10^{-9} with the accuracy 10^{-7} and for the second-order effective potential the phase shifts at the energies 10^{-9} and 10^{-5} with the same accuracy. The results below are dependent on the choice of the initial guess for fitted parameters. We have taken into account only those solutions for which the second order theory improvements improves the results of the first order one. In the Table 5.2, we can see the improvement in the second order, it is, however, very subtle, as will become clear from the Fig. 5.1.

n_r	Unknown	Coul	1oET	2oET
0	-1.78957010	-0.49999998	-1.40491850	-1.42048050
1	-0.18565448	-0.12500000	-0.18645510	-0.18625204
2	-0.07126672	-0.05555555	-0.07133907	-0.07132258
3	-0.03750993	-0.03125000	-0.03752442	-0.03752118
4	-0.02310461	-0.02000000	-0.02310893	-0.02310796
5	-0.01564935	-0.01388888	-0.01565098	-0.01565061
6	-0.01129715	-0.01020408	-0.01129787	-0.01129771
7	-0.00853719	-0.00781250	-0.00853756	-0.00853747
8	-0.00667777	-0.00617283	-0.00667797	-0.00667792
9	-0.00536578	-0.00500000	-0.00536590	-0.00536587

Tab. 5.2: Effective theory approximation for the binding states of the potential H. n_r is the radial number of the bound state. Unknown stands for the original potential (pseudodata), 1oET and 2oET for the results of the 1st and 2nd order effective potential, respectively.

The following Tab. 5.3 summarizes the phase shifts for various energies.

En [Me V]	Uknown	Coul	1oET	2oET
10^{-4}	18.72665960	18.17201407	18.72663414	18.72664168
10^{-3}	18.38120341	16.74450201	18.38094568	18.38100262
10^{-2}	14.85567425	13.92872582	14.85533460	14.85540659
10^{-1}	9.33682364	8.33889168	9.33156062	9.33260559
10^{0}	4.69859834	3.80369495	4.66430571	4.66941660
10^{1}	1.92693564	1.48989800	1.90219690	1.90609018

Tab. 5.3: The phase shifts for the original potential H, for the 1st and 2nd order effective potential and for the Coulomb-like potential alone versus the energy.

Three figures follows. Figure 5.1 graphically represents the results of Tab. 5.2. Figure 5.2 shows the same results but for the accuracy of the pseudodata of 10^{-5}. Figure 5.3 plots the phase shifts from Tab. 5.3 as the function of energy.

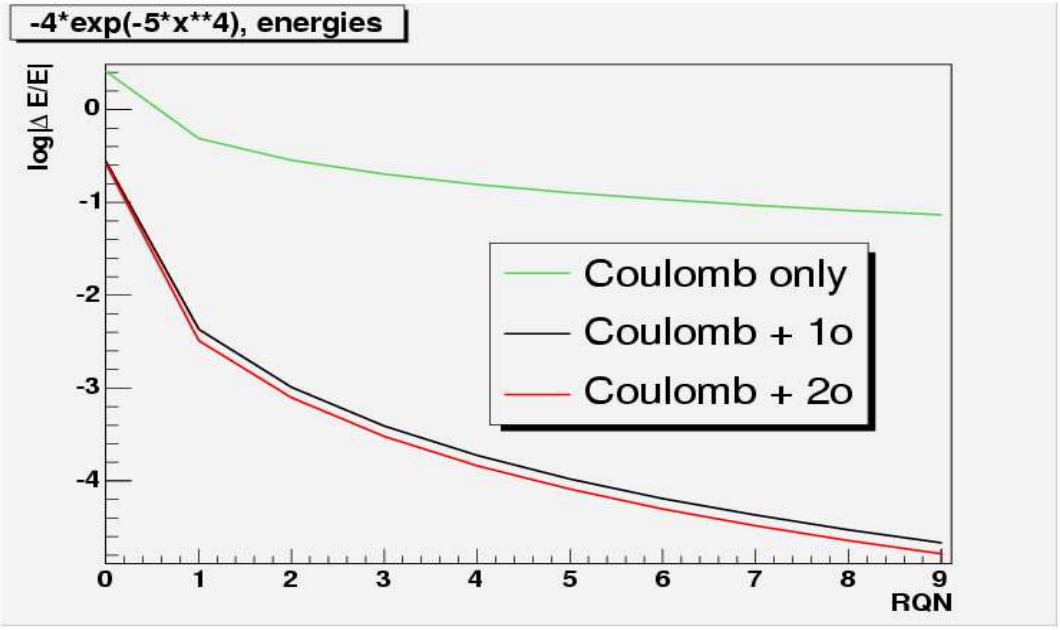

Fig. 5.1: The approximation of the binding energies by the 1st and 2nd order V_{eff}, respectively. The deviation $|\log(\Delta E/E)|$ is plotted, as in [40], where E is the exact energy and ΔE is the difference of the energy given by the respective V_{eff} and the exact one. For comparison, the purely Coulomb-like interaction is also included, which corresponds to $V_{eff} = 0$.

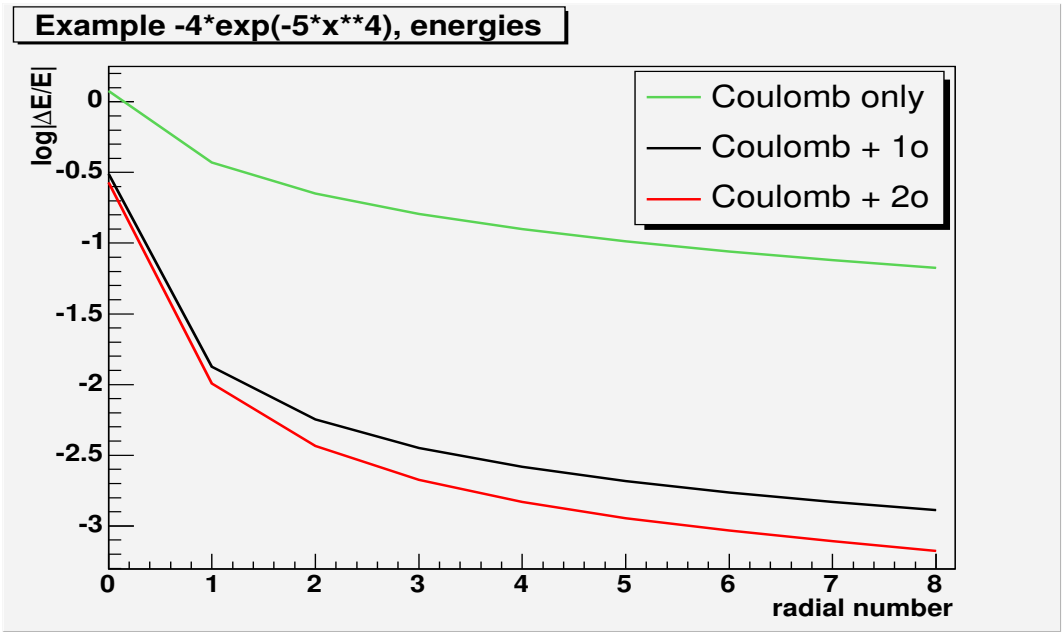

Fig. 5.2: The same as Fig. 5.1, but for accuracy of 10^{-5}.

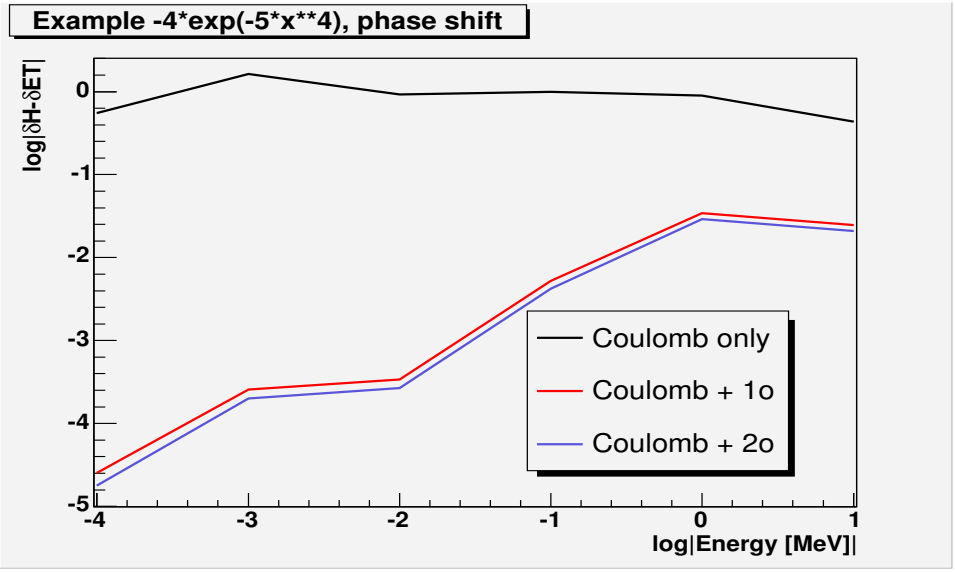

Fig. 5.3: The phase shifts for the original potential, for the 1st and 2nd order effective theory and for the purely Coulomb-like potential versus the energy.

Since to build the effective theory of potential does not mean to fit the shape of the potential, but to approximate the observables, we plot the shape of potentials comparing the effective and original ones. It is clear that they indeed differ. We can see that V_{eff} at the 1st and 2nd order are not significantly different, the second order term serves as a subtle correction to the first order one.

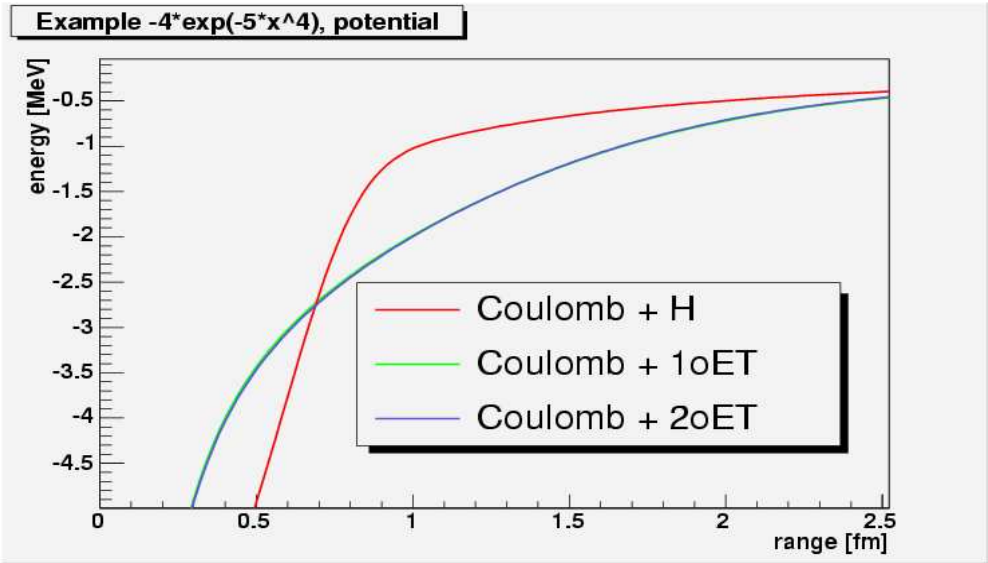

Fig. 5.4: The original and the 1st and 2nd-order effective potentials plotted as the function of the distance.

In the previous section, we have mentioned the dependence of the tuning of the 2nd-order effective theory on the initial guess. We have encounter this problem with each of the potentials, i.e., for each of them we found more than one pair of coefficients that satisfy the minimalizing condition. Each solution has to be checked to give meaningful results. We will illustrate this on the case of the potential A. Table 5.5 lists phase shifts for three pairs of coefficients, each for the different choice of initial guess (listed in Table 5.4). Phase shifts differ significantly for higher energies, i.e., each pair of coefficients approximate the original phase shifts differently. Thus we pick up the solution, for which there is the best agreement in the whole range of energies (not just for the two energies use din the fit). For the case given in tables below, the first pair is taken.

Guess: c & d_1	Result: c & d_1
G1: -0.12 & 0.10	-0.117032 & 0.038955
G2: 3.00 & 2.00	1.661150 & 1.779298
G3: -3.00 & 2.00	407.941836 & -148.661457

Tab. 5.4: Dependence of the result of the minimalization procedure (tuning the 2nd order effective theory with the low-energy phase shift) on the initial guess.

En [Me V]	Uknown	1oET	2oET(G1)	2oET(G2)	2oET(G3)
10^{-10}	18.84889056	18.84889056	18.84889056	18.84889056	18.84889057
10^{-9}	18.84745189	18.84745189	18.84745188	18.84745188	18.84745189
10^{-8}	18.84290239	18.84290239	18.84290239	18.84290239	18.84290242
10^{-7}	18.82851565	18.82851567	18.82851565	18.82851565	18.82851574
10^{-6}	18.78302203	18.78302209	18.78302202	18.78302202	18.78302229
10^{-5}	18.63919798	18.63919818	18.63919798	18.63919798	18.63919798
10^{-4}	18.18564008	18.18564069	18.18563990	18.18564009	18.18561249
10^{-3}	16.79366831	16.79366953	16.79366190	16.79366980	16.79255034
10^{-2}	14.12800493	14.12797881	14.12779570	14.12818491	14.08965031
10^{-1}	8.48491822	8.48496338	8.48365276	8.49681133	8.18738004
10^{0}	3.90681384	3.91740165	3.91367196	4.41669431	1.76765775
10^{1}	1.53388047	1.54051308	1.53992097	1.93145875	-7.57815087

Tab. 5.5: Dependence of the phase shifts on the energy for three different solutions for the coefficients of the 2nd order effective theory.

Regularization-related topics

In the general discussion of the effective theory we emphasized two features: the results of the effective theory of potential should not depend on the choice of smearing of δ-function;

the results should be approximately cut-off independent (when it is varied within physically reasonable bounds) .

Variation of the cut-off

We have chosen several values of the cut-off a (Table 5.6) and constructed for them the 1st order V_{eff}.

a	0.1	1.0	3.0	10.0
c/a	-46.157277	-1.626638	-0.306008	-0.049208

Tab. 5.6: Values of the cut-off a used in this subsection.

The following Fig. 5.5 presents the phase shifts as the function of the energy for each choice of the cut-off. As expected, we observe that the value $a = 1$ is the most suitable one. When the value of a is too large or too small, the accuracy of the effective approach worsens significantly, though the general trends are still visible.

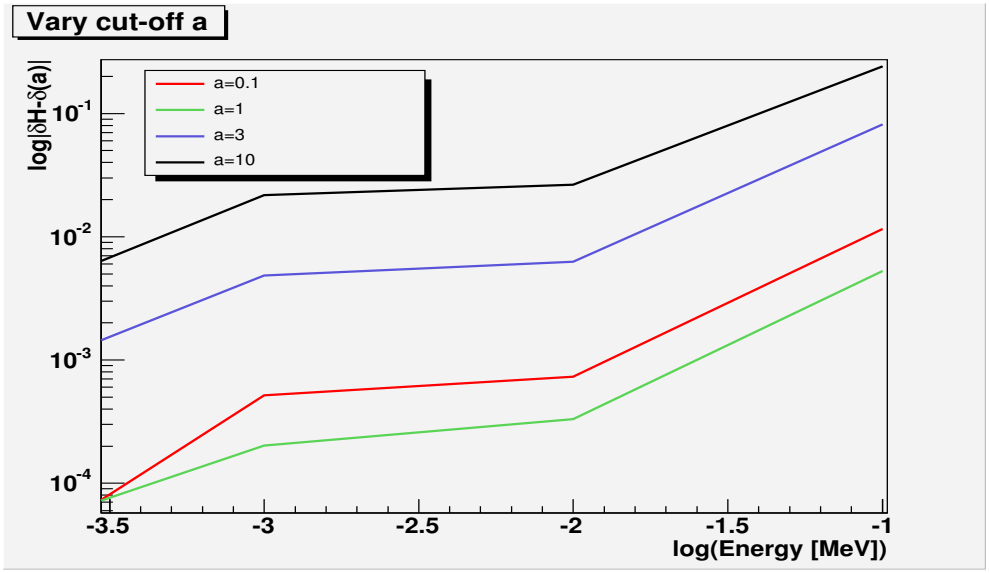

Fig. 5.5: Phase shifts as the function of the energy for various values of the cut-off a.

Smearing of delta function

The effective theory of potential states that the functional form of the smearing of δ-function is irrelevant. We will demonstrate this in the following way: we take a new shape of the smearing

$$\delta(\mathbf{x}) \equiv \frac{0.5}{\mathbf{x}^2 + 0.5^2}\,, \tag{5.3}$$

and the corresponding effective potential up to the second order

$$V_{eff}(r; c, d_1) = -\frac{\alpha}{r} + c \cdot \frac{0.5}{r^2 + 0.5^2} + d_1 \cdot \frac{-2 \cdot 0.5 \cdot (-r^4 + 2r^2 \cdot 0.5^2 + 3 \cdot 0.5^4)}{(r^2 + 0.5^2)^4}\,, \tag{5.4}$$

and fit again the new coefficients in the usual way on the data from the potential H. In the 1st order we get

$$c = -1.0108473\,, \qquad d_1 = 0\,, \tag{5.5}$$

and for the 2nd order:

$$c = -0.4854275\,, \qquad d_1 = 0.1341437\,. \tag{5.6}$$

The following Fig. 5.6 and Tab. 5.7, to be compared with Fig. 5.3 and Tb. 5.3 for the old form of smearing, summarize the results.

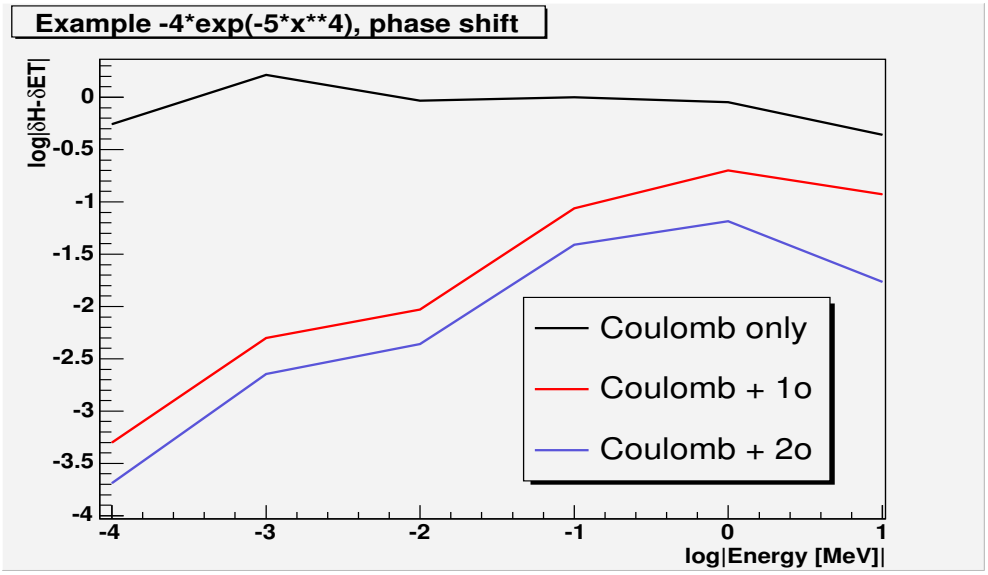

Fig. 5.6: Phase shifts as the function of the energy in the 1st and 2nd-order effective theory for δ-function smeared by (5.3)

En [Me V]	Uknown	Coul	1oET	2oET
10^{-4}	18.72665960	18.17201407	18.72616093	18.72645554
10^{-3}	18.38120341	16.74450201	18.37620734	18.37894549
10^{-2}	14.85567425	13.92872582	14.84629847	14.85131564
10^{-1}	9.33682364	8.33889168	9.25002229	9.29776588
10^{0}	4.69859834	3.80369495	4.49883756	4.63351624
10^{1}	1.92693564	1.48989800	1.80901435	1.94417485

Tab. 5.7: Phase shifts for the new form of the smeared δ-function (5.3).

It is clear from this simple check that the effective theory exhibits the same trends as in the case of previous form of smearing was used. It certainly does not prove the statement about the independence on the smearing, but it provides an illustrative example. One sees

in particular, that the form of smearing affects more the 1st order results, than the 2nd order ones, in accordance with the expectation that the cut-off and smearing dependence become weaker when higher orders of effective theory are included.

Finally, we show in Fig. 5.7 the shapes of the 2nd order potentials for new and old form of δ-function smearing.

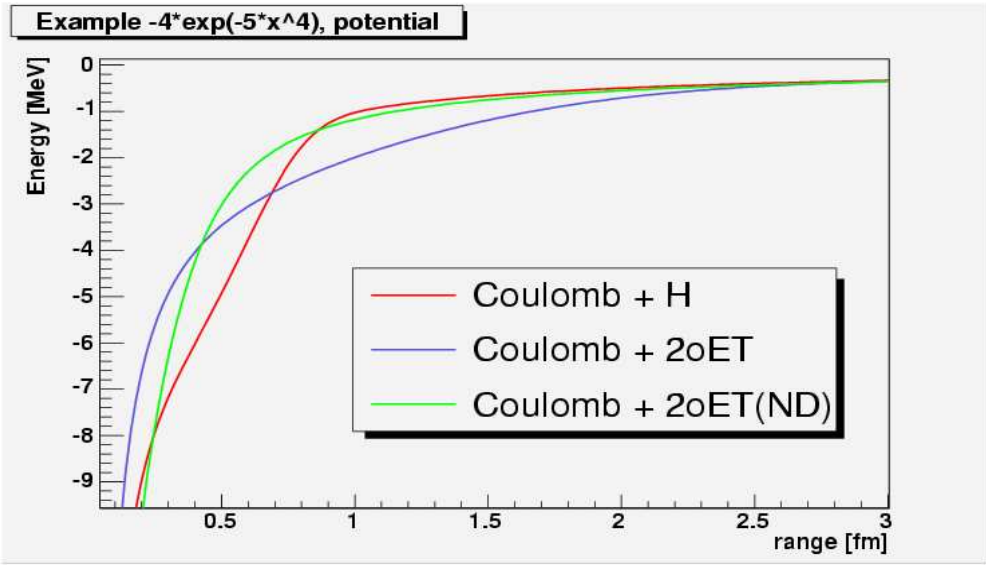

Fig. 5.7: Shapes of the 2nd order V_{eff} for two different forms of the δ-function smearing given by (4.8) and (5.3). The original potential H with the Coulomb-like interaction is also shown.

Purely short-range potential

We have applied the effective theory to a case of the purely short-range potential, i.e., we have switched off the long-range Coulomb-like interaction. As an example, we show the results for the potential:

$$V_S = -25\exp(-2\,x^4)\,, \tag{5.7}$$

where the depth of this potential has been chosen to give two bound energy levels. The purely short-range potential allows us to test the estimate of the number of bound states from the value of the phase shift obtained through the variable phase method (Appendix A).

Fitting the coefficients of V_{eff} to the data in the usual way, we get for the first order

$$c = -9.463889\,, \qquad d_1 = 0 \tag{5.8}$$

and for the second order

$$c = -8.994282\,, \qquad d_1 = 0.458574\,. \tag{5.9}$$

We summarize results for phase shifts in the Table below, the due to the 2nd-order is apparent.

En [Me V]	Uknown	1oET	2oET
10^{-5}	6.27853517	6.27853514	6.27853507
10^{-4}	6.26848035	6.26847964	6.26847959
10^{-3}	6.23668660	6.23666400	6.23667293
10^{-2}	6.13621992	6.13551257	6.13582242
10^{-1}	5.82082467	5.80060431	5.80888922
10^{0}	4.88816443	4.59260910	4.66771809
10^{1}	3.06274353	2.32295234	2.40771624

Tab. 5.8: 1st and 2nd order phase shifts for several energies for the purely short-range potential.

Finally, we present the results for the bound state energies, which are not fitted at all, but calculated from the effective potential fitted to low energy phase shifts. As follows [40] from the effective theory of potential, the higher energy level is described much better than the lower one.

n_r	Unknown	1oET	2oET
0	-16.806971	-5.335944	-5.857121
1	-3.060035	-1.329970	-1.463283

Tab. 5.9: Bound state energies of the purely short-range potential and the corresponding 1st and 2nd order V_{eff}.

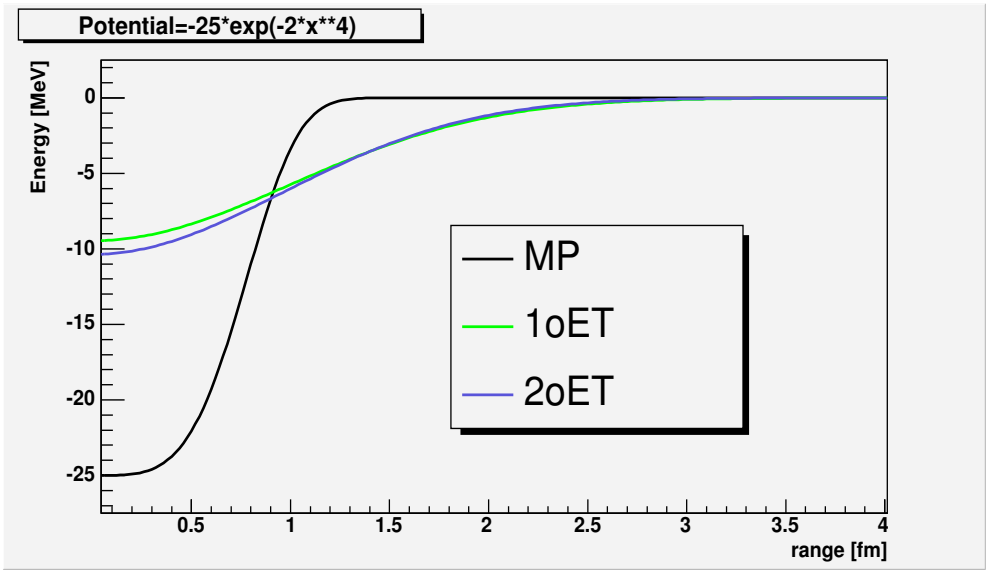

Fig. 5.8: The shape of the purely short-range potential and the corresponding 1st and 2nd order V_{eff} for the case without the long-range interaction.

For completeness, we present again the shape of the model potential H together with the 1st and 2nd order V_{eff} for the switched off long-range interaction. In this case, the difference between 1st and 2nd order V_{eff} is clearly visible.

5.3 Discussion

The goal of our numerical calculations was to learn the effective technique in quantum mechanical framework and verify results presented in [40]. We succeeded in building up the effective theory for a broad range of potentials up to second order. As follows from the theory, accuracy of the approximation of binding energies and phase shifts is increasing when the second order effective potential is included. However, this trend is not so pronounced as presented in [40]. This can partially be due to still insufficient accuracy of numerical algorithms we adopted, in particular of the minimalization procedures. Unfortunately, details of the numerics (and even the model short-range potential used to produce the pseudodata) are not published in [40]. Hence we could not consider exactly the same problem and compare the results just by putting them side-by-side.

Our next result, which has to be mentioned, is the existence of multiple solutions for given accuracy. This has not been mentioned in [40]. It brings up quite difficult problems with matching the coefficients of the effective theory. In particular, it is rather interesting that using very high precision solves at least partially these problems, at least partially. On the other hand, it implies that if applied to real data, typically measured with much lower precision, the effective theory of potential might face severe problems in finding an optimal, or even some reasonable, set of low energy constants. Let us point out that in a field theory the situation is different. There the global fits of the low energy parameters are usually not done, since it is possible (at least at lowest orders) to pick up different processes which are selectively sensitive to particular constants.

We have also verified the dependence of our approach on the variation of the cut-off and on the form of the smeared delta function. Our results are in good agreement with [40]. Finally, we applied the effective theory approach to the purely short-range potential.

As our numerical calculations were gradually building up, it was necessary to develop several numerical codes and implement various algorithms. Each tool was tested on some artificial exactly solvable problem. We succeeded in tuning up the code to reach a desired level of accuracy.

6 Conclusions

The effective approach is an universal way to treat theories with multiple scales. For the nucleon-nucleon sector two main scales exist, first one is connected to the mass of the nucleon, the second one to the mass of lightest meson, the pion. We have reviewed how the Lagrangian formalism for such a system is built up and how to deal with the infinities that arise. Three regularization schemes were considered: cut-off scheme, dimensional regularization with minimal subtraction scheme and, finally, dimensional regularization with power-law divergence subtraction. The last one was invented specifically to solve problems for the NN interaction in s-channels, where additional scales exist. The whole problem can be solved in more general approach, using the language of the renormalization group equation. We found this approach very interesting, however, it will need more effort and study to understand it completely.

The effective theory of potential is an interesting application of the effective theories to quantum mechanical framework. It should be stressed, that it is not an attempt to fit some particular unknown potential to the highest precision, but an universal method how to simulate the unknown short-range interaction. In our study of this approach we have encountered some numerical difficulties that have not been yet completely understood, it will need more time and effort to explain them in a satisfactory way. The first of this problems is multiple solutions of our minimalization procedure. There is also a question on whether the effective theory of potential does not require knowledge of experimental data to too high precision, this would jeopardize its application to real physical problems.

We would like to point out once more that we consider this study to be our first step into rather broad and involved field of EFT formalism and its application to hadronic processes. The acquired knowledge and experience with numerics should lead us in further learning of this technique and in future applications to more realistic problems.

Preliminary results of this thesis were presented on 29^{th} of March, 2005, at the regular seminar of the Institute of Experimental and Applied Physics of Czech Technical University in Prague.

Acknowledgement

First of all, I would like to thank my supervisor Jiří Adam for numerous interesting discussions and for invaluable help in preparing this thesis.

I am obliged to Miloslav Sotona, for providing his code calculating the binding energies and for helping us to modify it for our needs.

I also express my gratitude to my fellow students for providing very good and stimulating atmosphere.

Finally, I am sincerely grateful to my parents and to Andrea for their help, patience and encouragement during my study.

Appendix A

The Variable Phase Method

The Variable Phase Method is an important alternative to the straightforward integration of the radial Schrödinger equation for computing phase shifts. This method will be briefly described now, more can be found in [41, 42]. For the sake of simplicity only the s-wave will be discussed.

For a given potential $V(r)$, a new potential $V_\rho(r)$ can be defined by truncating the original potential at $r = \rho$, i. e.:

$$V_\rho(r) = \begin{cases} V(r) & r \leq \rho \\ 0 & r > \rho \end{cases} \tag{A.1}$$

Let us denote the regular solution for the actual potential V by $\phi(r)$ and appropriate phase shift by δ, for the truncated potential V_ρ they are $\phi_\rho(r)$ and $\delta(\rho)$, respectively. For $\rho = 0$ truncation leaves no potential at all and phase shift becomes an integer multiple of π. Such ambiguity can be removed by requiring $\delta(0) = 0$. On the other hand, it can be proved, that the following limit holds:

$$\lim_{\rho\to\infty} \delta(\rho) = \delta\,. \tag{A.2}$$

For any value of ρ the two functions $\phi(r)$ and $\phi_\rho(r)$ are constrained by the same boundary conditions at $r = 0$. They satisfy the same differential equation for $0 \leq r \leq \rho$, i.e.,

$$\phi(r) \equiv \phi_\rho(r),\ r \in \langle 0, \rho\rangle\ . \tag{A.3}$$

Beyond this interval, the two solutions differ. In particular, $\phi_\rho(r)$ satisfies the free radial equation and therefore can be written as

$$\phi_\rho(r) = C(\rho)\sin[pr + \delta(\rho)]\,, \tag{A.4}$$

where $p = (2mE)^{1/2}$ is the momentum of the particle and $C(\rho)$ is some ρ-dependent normalization coefficient.

Both solutions are continuous and identical for $r \in \langle 0, \rho\rangle$, thus $\phi(r)$ can be equated to $\phi_\rho(r)$ at $r = \rho$:

$$\phi(\rho) = \mathrm{C}(\rho)\sin[p\rho + \delta(\rho)]\,. \tag{A.5}$$

Derivative can be equated in a similar way. This holds for any choice of ρ, hence ρ can be replaced by r. Now, equations (A.5) and the analogous one for the derivative are inserted into the Schrödinger equation. This leads to the following differential equation for $\delta(r)$:

$$\delta'(r) = -\frac{2mV(r)}{p}\,\sin^2[pr + \delta(r)]\,. \tag{A.6}$$

Solution of this equation yields a finite phase shift only for potentials

$$V(r) \approx O(r^{-1-\epsilon}),\ r \to \infty\,. \tag{A.7}$$

Now recall the Levinson's theorem:

Levinson's theorem. For any spherical potential the phase shift $\delta_l(p)$ satisfies

$$\delta_l(0) - \delta_l(\infty) = n_l\pi\,, \tag{A.8}$$

where n_l denotes the number of bound states of an angular momentum l. This might have to be modified for the s-wave. If the s-wave Jost function vanishes at threshold $f_0(0) = 0$, the equation (A.8) becomes

$$\delta_0(0) - \delta_0(\infty) = (n_0 + 1/2)\pi\,. \tag{A.9}$$

The particular values of $\delta_l(x)$ still posses the modulo-π ambiguity, which can be eliminated by requirement

$$\delta_l(\infty) = 0\,. \tag{A.10}$$

It is proven in [42], that using the variable phase method in the limit of zero energy leads to (recall (A.10)):

$$\delta_l^{VPhM}(0) = n_l\pi\,, \tag{A.11}$$

and, analogously, for the exceptional s-wave case:

$$\delta_0^{VPhM}(0) = (n_0 + 1/2)\pi\,. \tag{A.12}$$

Hence the phase shifts obtained from the variable phase method carry an information about the number of bound states.

Appendix B

Numerov method

The Numerov algorithm is a numerical method for solving the ordinary second-order differential equation of the form

$$\frac{d^2y}{dx^2} = U(x) + V(x)y\,, \tag{B.1}$$

The well-known example of this class of equations is the Schrödinger equation

$$\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar}\left(V(x) - E\right)\psi\,. \tag{B.2}$$

The centralized differential scheme yields

$$y_{n+1} - 2y_n + y_{n+1} \approx 2\left(\frac{h^2}{2}\frac{d^2y}{dx^2} + \frac{h^4}{4!}\frac{d^4y}{dx^4} + \mathcal{O}(h^6)\right)\,, \tag{B.3}$$

where $y_n = y(x_n)$ and x_n are equidistant points separated by the step h. Putting $F = U(x) + V(x)y$ and using (B.1) one gets

$$y_{n+1} = 2y_n - y_{n-1} + h^2F_n + \frac{h^4}{12}\left.\frac{d^2F}{dx^2}\right|_n + \mathcal{O}(h^6)\,. \tag{B.4}$$

Expressing the second derivative of F with the help of (B.3) gives

$$y_{n+1} = \frac{2y_n - y_{n-1} + \frac{h^2}{12}\left(U_{n+1} + 10F_n + F_{n-1}\right)}{\left(1 - \frac{V_{n+1}h^2}{12}\right)} + \mathcal{O}(h^6)\,. \tag{B.5}$$

The error of this method is $\mathcal{O}(h^6)$ with one evaluation of U and V per step, while the Runge-Kutta algorithm needs at least six evaluations per step to achieve the same error level.

The recurrence (B.5) is initialized by the values of y_0 and y_1. For the Schrödinger equation the value of y_0 is known from the boundary condition of wave function at the origin. An inefficient method to get y_1 is to take the Taylor approximation, however, this causes a loss of precision. The better method is to determine y_1 from the values of F_0, F_1 and F_2

$$y_1 = y_0 + hy_0' + h^2\left(aF_0 + bF_1 + cF_2\right)\,, \tag{B.6}$$

where coefficients a, b and c are from the Taylor expansion of F_1 and F_2 at the origin. This lead to

$$y_1 = y_0 + hy_0' + \frac{h^2}{24}\left(7F_0 + 6F_1 - F_2\right) + \mathcal{O}(h^5)\,. \tag{B.7}$$

The standard Numerov procedure above for gives y_2

$$y_2 = \frac{2y_1 - y_0 + \frac{h^2}{12}\left(U_2 + 10F_1 + F_0\right)}{\left(1 - \frac{V_2 h^2}{12}\right)} + \mathcal{O}(h^6)\,, \tag{B.8}$$

This is the system of equations for y_1 and y_2, from which y_1 can be determined

$$y_1 = \frac{y_0\left(1 - \frac{V_2 h^2}{24}\right) + h y_0'\left(1 - \frac{V_2 h^2}{12}\right) + \frac{h^2}{24}\left(7F_0 + 6U_1 - U_2\right) - \frac{h^4 V_2}{36}\left(F_0 + 2U_1\right)}{1 - \frac{V_1 h^2}{4} + \frac{V_1 V_2 h^4}{18}}\,. \tag{B.9}$$

The accuracy of this estimate of y_1 is $\mathcal{O}(h^5)$.

Appendix C

Numerical techniques

This appendix describes some aspects of our numerical calculations. First two parts describe calculations of binding energies and phase shifts, respectively. The last part focuses on the tuning of the effective theory (i.e., fitting of the low energy constants). It is in this part, where we had to face most numerical difficulties.

C.1 Binding energies

Binding energies were computed by matching the inward and outward solutions of the Schrödinger equation at the classical turning point. We have used and adapted the numerical code (Fortran 77) developed by M. Sotona. This complex code is suited not only for computing binding energies, it also computes wave functions, can treat confining potentials, e.m. interactions with smeared charges etc. We had to adapt this code for our purposes. Since some of the potentials we considered have many closely clustered energy levels, we had to improve the part of the algorithm which directs the code from some guess of the energy to an improved one. A special care was necessary to ensure that no energy level is missed and that the code does not loose its way when energy levels are rather close (i.e., that it does not jump, trying to improve the guess, from one level to another).

To numerically integrate the Schrödinger equation for a given energy the code employs the Numerov algorithm (Appendix B). The asymptotic wave function is computed from the Whittaker functions. The binding energies were computed with the precision of 10^{-9}.

Such a high precision could not be achieved for the purely short-range potential with a shallow-bound state due to the numerical limitation of the code: the employed Numerov algorithm is not adaptive, it uses the fixed integration step. Since in this case the classical turning point is rather far from the origin, too many integration steps are needed to reach it and numerical errors accumulate.

C.2 Phase shift

Phase shifts were computed either (see (i), (ii) and (iv) below) with the variable phase method (appendix A) or (iii) matching the logarithmic derivatives of the outward and inward solutions of the Schrödinger equation. Several numerical algorithms were employed:

(i) the standard Runge-Kutta method of fourth order with a constant step
(ii) the standard Runge-Kutta method of fourth order with a variable step
(iii) the Numerov method
(iv) the predictor-corrector method – Gear's formula

Our experience with these algorithms is briefly summarized below:

Ad *(i)*: We used our own implementation (in C), no external library was used. It was usually necessary to take more than 10^6 steps to reach the accuracy 10^{-5}. This large number of steps was due to the singular behavior of the Coulomb-like term in the potential. The singularity itself was treated by shifting the origin $0 \to 10^{-30}$. It would be possible to

make more sophisticated correction, but the adopted approach was sufficiently accurate. This method is rather time consuming.

Ad *(ii)*: The routine DIVPRK from the IMSL library [44] was used. This method is fast and accurate, but its use is in general not recommended [43] when very high accuracy is required. We used it just for comparing with other methods.

Ad *(iii)*: The phase shift obtained by this method has the modulo-π ambiguity, thus it has been used only to check other methods. Nevertheless, it appeared to be more accurate than the method above.

Ad *(iv)*: The IMSL [44] routine DIVPAG was used. This method is fast and rather accurate. The results presented in this thesis were obtained in this way. The phase shifts were computed with relative accuracy of 10^{-9}.

C.3 Tuning the effective theory

C.3.1 First order

As suggested by the effective theory, the coefficient c was fitted to the phase shift at rather small energy $E = 10^{-9}$. Phase shifts are matched with the accuracy of 10^{-7}. To determine c we used a very simple algorithm of splitting intervals. On average, ten to twenty iterations were necessary.

C.3.2 Second order

One parameter fit of a single parameter c, described above, was rather trivial. Unfortunately, to fit two-parameters of the second order effective potential (5.1) was numerically rather difficult and some unforeseen problems have been faced. Similar to [40], we used two phase shifts for energies 10^{-9} and 10^{-5}, respectively, to fit the two parameters c and d_1. Thus, we had to solve a minimalization problem for two variables, minimalizing the positive quantity

$$f(c, d_1) = [\delta_{PD}(10^{-9}) - \delta(c, d_1; 10^{-9})]^2 + [\delta_{PD}(10^{-5}) - \delta(c, d_1; 10^{-5})]^2 \,, \qquad (C.1)$$

where the subscript PD stands for pseudoexperimental data.

We have implemented three different methods to solve the problem:

(a) the simplex algorithm

(b) the Metropolis algorithm

(c) the Levenberg-Marquardt algorithm with finite-difference Jacobian

Again, our experiences are briefly summarized:

Ad *(a)*: The subroutine AMOEBA from [43], implementing the downhill simplex method, was used. This method is suited only for searching for a local minimum of multidimensional function. It was not easy to achieve the precision we required. This method was used along with the Runge-Kutta method with constant step (to get the phase shifts). It resulted in very slow and often not terminating code.

Ad *(b)*: The subroutine AMEBSA from [43], combining the Metropolis algorithm with the modified amoeba-like one. This code depends on too many input parameters and it was not easy to understand the effect they have on the run of the code.

Ad *(c)*: The IMSL subroutine DUNLSF was used. Compared to previous two, it was easy to handle and efficient.

At first, we used the method (a). It was possible to use only the precision of 10^{-5} for comparing the phase shifts. The necessary input of all implemented algorithms is an initial guess for the fitted parameters. We have assumed that the first coefficient c has in the second order of the effective theory roughly, but not exactly, the same value as in the first-order. Thus, the guess value was roughly, up to ten percent, equal to the first-order value of c. The guess of the second parameter was $d_1 \sim -0.1\ c$. This was supported by the fact, that coefficients of the effective potential should roughly correspond to those from the Fourier transform of the potential (4.6). This was confirmed by other authors making a similar analysis [40, 45]. Unfortunately, results of the fit were initial guess dependent. That is, we were getting different pairs of coefficients, approximating phase shift values with given accuracy, by changing the initial guess for the minimalizing code.

This was very unexpected result, authors of a similar analysis [40] do not mention such ambiguity. At the same time, all found values of the coefficients were defining the effective potential, which approximated the pseudodata better than the first-order one. Our minimalization problem obviously has many local minima and the adopted techniques could not find an absolute one.

We made two attempts to cure this. First, we employed matching algorithm with higher accuracy 10^{-7} and better algorithms (b) and (c). We were not able to implement the method (b) for our purposes, so we dropped it and used the method (c) instead. This method was efficient, fast and matched the phase shifts with the required accuracy.

Higher accuracy partially solves the ambiguity. The number of devious coefficients significantly decreased, however, the solution was still not unique. This experience nevertheless explains, why other authors (e.g., Lepage) calculated their energies and phase shifts to very high precision (10 and more digits), which is an accuracy that would be impossible to achieve experimentally for any realistic physical system.

Next, we tried to increase the interval between the two energies used for the fit, taking $E_1 = 10^{-9}$ and $E_2 = 10^{-2}$. This, however, does not give any significant improvement of the uniqueness of the fit. We were still finding many devious pairs of coefficients.

We also tried to fix the coefficient c to the value found in the first order of the effective theory. Then only one coefficient d_1 has to be determined in the second order of tuning. This idea is motivated by the fact that d_1 is connected with a power of energy, while c is not. Thus, the observable at the energy E_1 will be much less influenced by d_1, than the one at higher energy E_2. Nevertheless, this approach is less accurate that the two parameter fit at the 2nd order.

At the end, we decided to choose energies $E_1 = 10^{-9}$ and $E_2 = 10^{-5}$ ([40] used very similar values), the precision 10^{-7} and minimalization method (c).

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