Kanonické transformace $\quad (\vec q,\vec p) \to (\vec Q,\vec P)$ na $\Gamma \quad \left.\begin{array}{l} Q_i = Q_i(\vec q,\vec p,t) \\ P_i = P_i(\vec q,\vec p,t)\end{array}\right\} \in C^{(2)}(\Gamma\times\mathbb{R}) \quad \left|\left(\frac{\partial(\vec Q,\vec P)}{\partial(\vec q,\vec p)}\right)\right| \neq 0$

Jacobiho Matice

$\Updownarrow$

$\{Q_i,Q_j\} = 0 = \{P_i,P_j\} \quad \{Q_i,P_j\} = \delta_{ij} \quad \forall i,j \in \hat{s}$

Vytvořující funkce kanonické transformace

1) 1. druhu $\quad F_1 = F_1(\vec q,\vec Q,t) \qquad dF_1 = p_i dq_i - P_j dQ_j + \underbrace{(K-H)}_{\frac{\partial F_1}{\partial t}}dt$

$$\boxed{\frac{\partial F_1}{\partial q_i} = p_i(\vec q,\vec Q) \quad \frac{\partial F_1}{\partial Q_j} = -P_j(\vec q,\vec Q)}$$

$K = H + \frac{\partial F_i}{\partial t} \quad i=1,2$

2) 2. druhu $\quad F_2 = F_2(\vec q,\underline{\underline{\vec P}},t) = F_1 - Q_j\cdot(-P_j) \qquad dF_2 = p_i dq_i + Q_j dP_j + (K-H)dt$

$$\boxed{\frac{\partial F_2}{\partial q_i} = p_i(\vec q,\vec P) \quad \frac{\partial F_2}{\partial P_j} = Q_j(\vec q,\vec P)}$$

19) Najděte kanonické transformace určené vytvořujícími funkcemi

$F_2 = \sum_1^{\hat s} q_k P_k \qquad p_i = \frac{\partial F_2}{\partial q_i} = \delta_{ik}P_k = P_i \qquad Q_j = \frac{\partial F_2}{\partial P_j} = q_k\delta_{jk} = q_j$ $\quad$ Identická Tr.

$F_2 = \sum f_k(\vec q,t)P_k \qquad p_i = \frac{\partial F_2}{\partial q_i} = \frac{\partial f_k}{\partial q_i}P_k \qquad Q_j = \frac{\partial F_2}{\partial P_j} = f_k\delta_{jk} = f_j(\vec q,t)$ $\quad$ Rozšířená bodová transformace $(M\to\Gamma)$

$F_1 = \sum q_k Q_k \qquad p_i = \frac{\partial F_1}{\partial q_i} = \delta_{ik}Q_k = Q_i \qquad P_j = -\frac{\partial F_1}{\partial Q_j} = -q_k\delta_{jk} = -q_j$ $\quad$ Prohození $\vec q$ a $\vec p$

20) Ukažte, že transformace $\underline{Q_j = p_j} \quad P_j = -q_j \quad \forall j\in\hat s$ je kanonická

- určíme kterou z F_i hledat
- závisí na čase?
- Napsat rce.

$\Leftrightarrow$ ① $\{Q_i,Q_j\} = \{p_i,p_j\} = 0$

$\{P_i,P_j\} = \{-q_i,-q_j\} = 0$

$\{Q_i,P_j\} = \{p_i,-q_j\} = \{q_j,p_i\} = \delta_{ji}$ ✓

$\Rightarrow$ F_2 nejde $p_i(\vec q,\vec P)$

② Najdeme $F_1(\vec q,\vec Q,t)$

$\boxed{\begin{array}{l} i=1,\dots,s \\ j=1,\dots,s\end{array}} \quad \frac{\partial F_1}{\partial q_i} = p_i(\vec q,\vec Q) = Q_i \qquad \frac{\partial F_1}{\partial Q_j} = -P_j(\vec q,\vec Q) = +q_j$

$i=1 \quad \frac{\partial F_1}{\partial q_1} = Q_1 \to F_1(\vec q,\vec Q) = \int Q_1 dq_1 + C_1(q_2,\dots,q_s,\vec Q) = Q_1q_1 + C_1(q_2,\dots,q_s,\vec Q)$

$i=2 \quad \frac{\partial F_1}{\partial q_2} = Q_2 \qquad \frac{\partial(Q_1q_1+C_1)}{\partial q_2} = Q_2 \qquad \frac{\partial C_1}{\partial q_2} = Q_2 \to C_1(q_2,\dots,q_s,\vec Q) = \underbrace{\int Q_2dq_2}_{Q_2q_2} + C_2(q_3,\dots,q_s,\vec Q)$

$\vdots$

$i=s \quad \frac{\partial F_1}{\partial q_s} = Q_s \qquad \frac{\partial C_{s-1}}{\partial q_s} = Q_s \to C_{s-1}(q_s,\vec Q) = \int Q_s dq_s + C_s(\vec Q) = Q_sq_s + C_s(\vec Q)$

$F_1(\vec q,\vec Q) = \sum q_kQ_k + C_s(\vec Q)$

$\forall j\in\hat s \quad \frac{\partial F_1}{\partial Q_j} = q_j \qquad \cancel{q_k\delta_{kj}} + \frac{\partial C_s}{\partial Q_j} = \cancel{q_j} \Rightarrow \frac{\partial C_s}{\partial Q_j} = 0 \Rightarrow \underline{C_s = konst} \Rightarrow \boxed{F_1(\vec q,\vec Q) = \sum q_kQ_k} + \underbrace{\cancel{C_s}}_{=0}$

③ $dF_1 = p_i dq_i - P_i dQ_i + \underbrace{(K-H)}_{\frac{\partial F_1}{\partial t}=0}dt = Q_idq_i - (-q_i)dQ_i = Q_idq_i + q_idQ_i = d(Q_iq_i)$

$\Rightarrow \underline{F_1 = Q_iq_i + \cancel{Const}}$ $\quad$ 0

21) Najděte transformaci defiovanou vytvořující funkcí $F_2(\overbrace{x_1,x_2,x_3}^{\vec q},\overbrace{P_R,P_\varphi,P_z}^{\vec P}) = \sqrt{x_1^2+x_2^2}\,P_R + \operatorname{arctg}\left(\frac{x_2}{x_1}\right)P_\varphi + x_3P_z$

$\vec Q = (R,\varphi,z) \quad Q_i = \frac{\partial F_2}{\partial P_i} \qquad p_i = \frac{\partial F_2}{\partial q_i}$

$R = \frac{\partial F_2}{\partial P_R} = \sqrt{x_1^2+x_2^2} \qquad R = \sqrt{1+\operatorname{tg}^2\varphi}\,x_1 = \frac{1}{\cos\varphi}x_1$

$\varphi = \frac{\partial F_2}{\partial P_\varphi} = \operatorname{arctg}\left(\frac{x_2}{x_1}\right) \to x_2 = x_1\operatorname{tg}\varphi$

$z = \frac{\partial F_2}{\partial P_z} = x_3$

cylindrické souřadnice $\boxed{\begin{array}{l} x_1 = R\cos\varphi \\ x_2 = R\sin\varphi \\ x_3 = z\end{array}}$

$p_1 = \frac{x_1}{\sqrt{x_1^2+x_2^2}}P_R + \overbrace{\frac{1}{1+\left(\frac{x_2}{x_1}\right)^2}\cdot\left(-\frac{x_2}{x_1^2}\right)}^{-\frac{x_2}{x_1^2+x_2^2}}P_\varphi = \cos\varphi\,P_R - \frac{\sin\varphi}{R}P_\varphi$

$p_2 = \frac{x_2}{\sqrt{x_1^2+x_2^2}}P_R + \frac{1}{1+\left(\frac{x_2}{x_1}\right)^2}\frac{1}{x_1}P_\varphi = \sin\varphi\,P_R + \frac{\cos\varphi}{R}P_\varphi$

$p_3 = P_z$

22) Ukažte, že transformace $Q = \operatorname{arctg}\left(\sqrt{km}\,\frac{q}{p}\right)$ $P = \frac{1}{2}\left(\sqrt{km}\,q^2 + \frac{p^2}{\sqrt{km}}\right)$ je kanonická.

Využijte ji k řešení harmonického oscilátoru $H = \frac{p^2}{2m} + \frac{1}{2}kq^2$

Najdeme $F_1(\vec{q}, \vec{Q}, t)$

$P = \frac{1}{2}\left(\sqrt{km}\,q^2 + \frac{km\,q^2 \cotg^2 Q}{\sqrt{km}}\right)$

$\frac{\partial F_1}{\partial q} = p = \sqrt{km}\,q \cotg Q \to F_1 = \int \sqrt{km}\,q \cotg Q\,dq + C(Q)$ $\quad P = \frac{1}{2}\sqrt{km}\,q^2\underbrace{(1 + \cotg^2 Q)}_{\frac{1}{\sin^2 Q}}$

$\tg Q = \sqrt{km}\,\frac{q}{p}$

$\frac{\partial F_1}{\partial Q} = -P = -\frac{1}{2}\sqrt{km}\,q^2 \frac{1}{\sin^2 Q}$ $\quad F_1 = \sqrt{km}\,\frac{q^2}{2}\cotg Q + C(Q)$

$\boxed{F_1 = \frac{1}{2}\sqrt{km}\,q^2 \cotg Q}$

$\cancel{\sqrt{km}\,\frac{q^2}{2}\left(\frac{-1}{\sin^2 Q}\right)} + \frac{\partial C}{\partial Q} = \cancel{-\frac{1}{2}\sqrt{km}\,q^2 \frac{1}{\sin^2 Q}} \Rightarrow \frac{\partial C}{\partial Q} = 0 \Rightarrow C = \text{konst.} \underset{\text{BÚNO}}{=} 0$

$q = q(Q,P)$ $\quad K(Q,P) = H(q(Q,P), p(Q,P))$

$p = p(Q,P)$ $\quad \frac{\partial F_1}{\partial t} = 0 \quad K = H + \frac{\partial F_1}{\partial t}$

$q = \pm\sqrt{\frac{2P}{\sqrt{km}}\sin^2 Q} = \sqrt{\frac{2P}{\sqrt{km}}}\sin Q$ $\quad *$

$p = q\sqrt{km}\cotg Q = \sqrt{2P\sqrt{km}}\cos Q$

$$H' = K = \frac{\left(\sqrt{2P\sqrt{km}}\cos Q\right)^2}{2m} + \frac{1}{2}k\left(\sqrt{\frac{2P}{\sqrt{km}}}\sin Q\right)^2 = \frac{\cancel{2}P\sqrt{km}}{\cancel{2}m}\cos^2 Q + \frac{k\cancel{2}P}{\cancel{2}\sqrt{km}}\sin^2 Q = \sqrt{\frac{k}{m}}P\underbrace{(\cos^2 Q + \sin^2 Q)}_{1} = \sqrt{\frac{k}{m}}P$$

$\dot{Q} = \frac{\partial K}{\partial P} = \sqrt{\frac{k}{m}} \Rightarrow Q(t) = \sqrt{\frac{k}{m}}t + Q_0$ $\quad q(t) = \sqrt{\frac{2P}{\sqrt{km}}}\sin\left(\sqrt{\frac{k}{m}}t + Q_0\right)$ $\quad P = \frac{E}{\omega}$ "energie"

$\dot{P} = -\frac{\partial K}{\partial Q} = 0 \Rightarrow P = \text{konst}$ $\quad p(t) = \sqrt{2P\sqrt{km}}\cos\left(\sqrt{\frac{k}{m}}t + Q_0\right)$ $\quad Q \ldots$ fáze

23) Pro která $\alpha, \beta \in \mathbb{R}$ je transformace kanonická $Q = q^\alpha \cos(\beta p)$ $\quad P = q^\alpha \sin(\beta p)$

Najděte příslušnou vytvořující funkci.

1) $\{Q,Q\} = 0 = \{P,P\}$ $\quad \{Q,P\} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = \alpha q^{\alpha-1}\cos(\beta p)\,q^\alpha \beta\cos(\beta p) - \left(-q^\alpha \beta \sin \beta p\,\alpha q^{\alpha-1}\sin \beta p\right)$

$= \alpha q^{2\alpha-1}\beta\left(\cos^2(\beta p) + \sin^2(\beta p)\right) = \alpha\beta q^{2\alpha-1} \overset{?}{=} 1$

$\forall q \Rightarrow 2\alpha - 1 = 0$

$\alpha \cdot \beta = 1$ $\quad \boxed{\beta = 2}$ $\quad \boxed{\alpha = \frac{1}{2}}$

Hledáme $F_1(q,Q)$ $\quad$ vyjádříme $dQ(q,p)$

2) $dF_1 = p\,dq - P\,dQ \overset{!}{=} p\,dq - q^\alpha \sin(\beta p)\left(\alpha q^{\alpha-1}\cos(\beta p)\,dq - q^\alpha \beta \sin(\beta p)\,dp\right) =$

$dQ = \frac{\partial Q}{\partial q}dq + \frac{\partial Q}{\partial p}dp + \cancel{\frac{\partial Q}{\partial t}dt} = \alpha q^{\alpha-1}\cos(\beta p)\,dq - q^\alpha \beta \sin(\beta p)\,dp$

3) $\frac{\partial F_1}{\partial q} = p$ $\quad \frac{\partial F_1}{\partial Q} = -P$ $\quad \frac{\partial^2 F_1}{\partial Q \partial q} = \frac{\partial^2 F_1}{\partial q \partial Q}$ $\Downarrow$ α, β

$$dF_1 = \underbrace{\left(p - \alpha q^{2\alpha-1}\overbrace{\sin(\beta p)\cos(\beta p)}^{\frac{1}{2}\sin(2\beta p)}\right)}_{\frac{\partial}{\partial p}\frac{\partial F_1}{\partial q}}dq + \underbrace{q^{2\alpha}\beta\sin^2(\beta p)}_{\frac{\partial}{\partial q}\frac{\partial F_1}{\partial p}}dp = \underbrace{\left(p - \frac{1}{4}\sin(4p)\right)}_{=\frac{\partial F_1}{\partial q}}dq + \underbrace{2q\sin^2(2p)}_{=\frac{\partial F_1}{\partial p}}dp$$

uzavřenost (exaktnost) $\quad \frac{\partial}{\partial p}\frac{\partial F_1}{\partial q} \overset{?}{=} \frac{\partial}{\partial q}\frac{\partial F_1}{\partial p}$

$$1 - \alpha q^{2\alpha-1}\frac{1}{2}2\beta\overbrace{\cos(2\beta p)}^{\cos^2(\beta p) - \sin^2(\beta p)} = 2\alpha q^{2\alpha-1}\beta\sin^2(\beta p)$$

$\boxed{Q = \sqrt{q}\cos(2p)}$

$1 - \alpha\beta q^{2\alpha-1}\underbrace{\left(\cos^2(\beta p) - \sin^2(\beta p) + 2\sin^2(\beta p)\right)}_{1} = 0$ $\quad 1 - \alpha\beta q^{2\alpha-1} = 0 \quad \forall q$ $\quad \boxed{\alpha = \frac{1}{2}, \ \beta = 2}$

$p = \frac{1}{2}\arccos\left(\frac{Q}{\sqrt{q}}\right)$

$F_1 = \int p - \frac{1}{4}\sin(4p)\,dq + C(p) = pq - \frac{q}{4}\underbrace{\sin(4p)}_{2\sin(2p)\cos(2p)} + \overset{=0}{C(p)}$

$$\boxed{F_1(q,Q) = \frac{q}{2}\arccos\left(\frac{Q}{\sqrt{q}}\right) - \frac{q}{2}\frac{Q}{\sqrt{q}}\sqrt{1 - \left(\frac{Q}{\sqrt{q}}\right)^2}}$$

$\frac{\partial F_1}{\partial p} = q - \frac{q}{\cancel{4}}\cancel{4}\cos(4p) + \frac{dC}{dp} = 2q\sin^2(2p) \Rightarrow \frac{dC}{dp} = 0 \Rightarrow \boxed{C = \text{konst}}$

$q\left(\underset{\cos^2(2p) + \sin^2(2p)}{1} - \cos^2(2p) + \sin^2(2p)\right) = q\left(2\sin^2(2p)\right)$