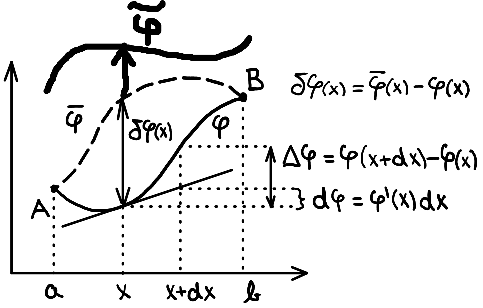
Variační počet body → křivky funkce → funkcionály



Křivka je zobrazení $\varphi : \langle a,b\rangle \subset \mathbb{R} \to \mathbb{R}$ třídy $C^{(n)}$, $n \in \mathbb{N}_0$

$C^{(n)}\langle a,b\rangle$ mn. všech křivek třídy n

$\|\varphi\| = \max\limits_{x\in\langle a,b\rangle}\{|\varphi(x)|, |\varphi'(x)|, \dots, |\varphi^{(n)}(x)|\}$ } normovaný vektorový prostor $\vec{C}^{(n)}\langle a,b\rangle$

$C^{(n)}_{(A,B)}\langle a,b\rangle = \{\varphi \in C^{(n)}\langle a,b\rangle \,|\, \varphi(a) = A \wedge \varphi(b) = B\}$ mn. křivek z A do B

$\vec{C}^{(n)}_{(0,0)}\langle a,b\rangle$ } normovaný afinní prostor

$$\boxed{\frac{d}{dx}(\delta\varphi) = \delta\left(\frac{d\varphi}{dx}\right)}$$

Buď $\varphi \in C^{(n)}_{(A,B)}\langle a,b\rangle$ pak pro lib. $\overline{\varphi} \in C^{(n)}_{(A,B)}\langle a,b\rangle$ nazýváme $\delta\varphi = \overline{\varphi} - \varphi \in \vec{C}^{(n)}_{(0,0)}\langle a,b\rangle$ variací křivky φ s pevnými konci.
$C^{(n)}\langle a,b\rangle$ $\vec{C}^{(n)}\langle a,b\rangle$ s volnými konci.

Funkcionál $I : C^{(n)}\langle a,b\rangle \to \mathbb{R}$ je diferencovatelný na křivce φ pokud existuje spojitý lineární funkcionál $\Phi : \vec{C}^{(n)}\langle a,b\rangle \to \mathbb{R}$ (nazývaný <u>variace I na křivce φ</u> značený $\delta I(\varphi)$) tak, že platí $\lim\limits_{\|\delta\varphi\|\to 0} \frac{I(\varphi+\delta\varphi) - I(\varphi) - \Phi[\delta\varphi]}{\|\delta\varphi\|} = 0$.

variace funkcionálu "=" lineární část přírustku

$$\Delta I = I(\varphi+\delta\varphi) - I(\varphi) = \underbrace{\Phi[\delta\varphi]}_{\delta I(\varphi)[\delta\varphi]} + \underbrace{\omega(\varphi,\delta\varphi)}_{\to 0 \text{ pro } \|\delta\varphi\|\to 0} \cdot \|\delta\varphi\|$$

existuje-li variace, lze ji najít takto

$$\delta I(\varphi)[\delta\varphi] = \frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} I(\varphi + \varepsilon\delta\varphi)$$

Funkcionál $I : C^{(n)}\langle a,b\rangle \to \mathbb{R}$ má na křivce φ stacionární hodnotu (φ je extremálou I) pokud $\delta I(\varphi) = 0$

ZLVP: Buď $g \in C\langle a,b\rangle$ pokud $\forall h \in C^{(n)}_{(0,0)}\langle a,b\rangle$ platí $\int_a^b g(x)\cdot h(x)\,dx = 0$ pak $g(x) = 0$ $\forall x \in \langle a,b\rangle$.

<u>Dále speciální případ funkcionálu</u>

Buď $F : \mathbb{R}^3 \to \mathbb{R}$ třídy $C^{(2)}$ pak funkcionál $J : C^{(1)}\langle a,b\rangle \to \mathbb{R}$, $J(\varphi) := \int_a^b F(x, \underbrace{\varphi(x)}_{y}, \underbrace{\varphi'(x)}_{y'})\,dx$ je diferencovatelný na $C^{(1)}\langle a,b\rangle$.

$$\delta J(\varphi)[\delta\varphi] = \int_a^b \left[\frac{\partial F}{\partial\varphi} - \frac{d}{dx}\left(\frac{\partial F}{\partial\varphi'}\right)\right]\delta\varphi\,dx + \underbrace{\left[\frac{\partial F}{\partial\varphi'}\delta\varphi\right]_a^b}_{=0 \text{ pro pevné konce}} \quad \begin{matrix}\delta\varphi(a)=0\\ \delta\varphi(b)=0\end{matrix}$$

$$\boxed{\delta\int_a^b F\,dx = \int_a^b \delta F\,dx}$$

Křivka $\varphi \in C^{(1)}_{(A,B)}\langle a,b\rangle$ je extremálou funkcionálu $J\big|_{C^{(1)}_{(A,B)}\langle a,b\rangle} \iff \frac{\partial F}{\partial\varphi}(x,\varphi(x),\varphi'(x)) - \frac{d}{dx}\left(\frac{\partial F}{\partial\varphi'}(x,\varphi(x),\varphi'(x))\right) = 0$ $\forall x \in \langle a,b\rangle$, $\varphi(a) = A$, $\varphi(b) = B$

Eulerova rovnice pro funkcionál J

ODR 2. řádu s okrajovými podmínkami

Zobecnění:

$\vec{\varphi} = \begin{pmatrix}\varphi_1\\ \vdots \\ \varphi_n\end{pmatrix}$ $\delta\varphi_i$ jsou nezávislé

1) $\vec{\varphi} : \langle a,b\rangle \subset \mathbb{R} \to \mathbb{R}^n \in C^{(1)}$ $\vec{\varphi}(a) = \vec{A}$ $\vec{\varphi}(b) = \vec{B}$

$F : \mathbb{R}^{2n+1} \to \mathbb{R} \in C^{(2)}$ $J(\vec{\varphi}) = \int_a^b F(x, \vec{\varphi}(x), \vec{\varphi}'(x))\,dx$ $\delta J(\vec{\varphi}) = 0 \wedge \delta\vec{\varphi}(a) = 0 = \delta\vec{\varphi}(b) \iff \forall i \in \hat{n}$ $\frac{\partial F}{\partial\varphi_i} - \frac{d}{dx}\left(\frac{\partial F}{\partial\varphi_i'}\right) = 0$ $\forall x \in \langle a,b\rangle$

2) $F = F(x, y, y', y'', \dots, y^{(k)})$ $\varphi \in C^{(k)}\langle a,b\rangle$ $J(\varphi) = \int_a^b F(x, \varphi(x), \varphi'(x), \dots, \varphi^{(k)}(x))\,dx \iff 0 = \frac{\partial F}{\partial\varphi} - \frac{d}{dx}\left(\frac{\partial F}{\partial\varphi'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial\varphi''}\right) - \frac{d^3}{dx^3}\left(\frac{\partial F}{\partial\varphi'''}\right) + \dots$

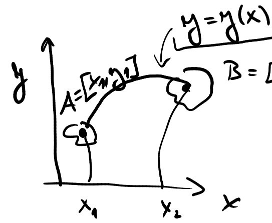
$\varphi : \langle a,b\rangle \to \mathbb{R}$

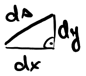
3) $\varphi : \mathbb{R}^4 \to \mathbb{R}$ $\in C^{(1)}$ $\varphi_\mu = \frac{\partial\varphi}{\partial x^\mu}$ $F = F(x^\mu, \varphi, \varphi_\mu)$ $0 = \frac{\partial F}{\partial\varphi} - \sum_{\mu=0}^{3} \frac{\partial}{\partial x^\mu}\frac{\partial F}{\partial\varphi_\mu}$

$\varphi = \varphi(x^\mu) = \varphi(x^0, x^1, x^2, x^3)$

1) Po jaké dráze mezi dvěma body ve svislé rovině xy se pohybuje včela, která se snaží dosáhnout cíle za nejkratší dobu, je-li její rychlost úměrná její výšce?

$v = ky \quad k>0,\ y>0$





čas $T[y(x)] = \int_0^{\bar s} \frac{ds}{v} = \int_{x_1}^{x_2} \underbrace{\frac{\sqrt{1+y'^2}}{ky}}_{F(y,y')} dx$

$y=y(x)$ je extremála $\iff$ Euler. rce $\boxed{\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0}$

$ds = \sqrt{(dx)^2+(dy)^2} = \sqrt{(1+(y')^2)(dx)^2} = \sqrt{1+y'^2}\,dx$

$dy = y'dx$

$\boxed{\frac{\partial F}{\partial x}=0} \Rightarrow \boxed{F - y'\frac{\partial F}{\partial y'} = Const.}$

$$0 = \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right)y' = \frac{d}{dx}\Big(\underbrace{F - y'\frac{\partial F}{\partial y'}}_{Const.}\Big) = \underbrace{\frac{\partial F}{\partial x}}_{=0} + \frac{\partial F}{\partial y}y' + \cancel{\frac{\partial F}{\partial y'}y''} - \cancel{y''\frac{\partial F}{\partial y'}} - y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)$$

$$C_1 = F - y'\frac{\partial F}{\partial y'} = \frac{\sqrt{1+y'^2}}{ky} - y'\frac{\cancel{2}y'}{\cancel{2}\sqrt{1+y'^2}}\frac{1}{ky} = \frac{1}{ky}\left(\frac{1+\cancel{y'^2}-\cancel{y'^2}}{\sqrt{1+y'^2}}\right) = \frac{1}{ky\sqrt{1+y'^2}}$$

$C_1^2k^2y^2(1+y'^2)=1$

$C_1^2k^2y^2y'^2 = 1 - C_1^2k^2y^2$

$\frac{dy}{dx} = y' = \sqrt{\frac{1-C_1^2k^2y^2}{C_1^2k^2y^2}}$

$$\int \frac{C_1 k y}{\sqrt{1-C_1^2k^2y^2}} \underbrace{y'dx}_{dy} = \int 1\,dx = x + C_2$$

$$-\frac{1}{C_1k}\sqrt{1-C_1^2k^2y^2} = x + C_2 \quad /^2$$

$$\boxed{\left(\frac{1}{C_1k}\right)^2 = y^2 + (x+C_2)^2}$$

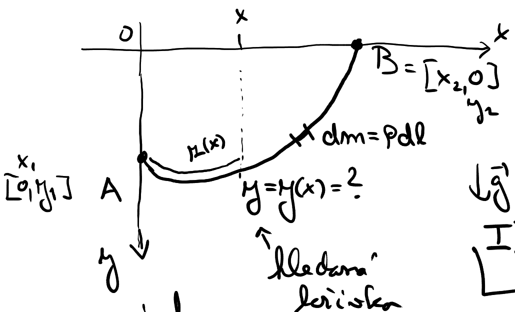
kružnica se středem v $[-C_2, 0]$

Konst. C_1 a C_2 určíme z $\boxed{y(x_1)=y_1,\ y(x_2)=y_2}$

2) Určete polohu těžkého homogenního vlákna pod vlivem tíže.

Rovnovážná poloha $\iff$ extrém $U = -\underbrace{Mg}_{Konst}\,y_T$ $\iff$ extrém $I[y(x)] = y_T$

těžiště vlákna



ρ ... lineární hustota

$\boxed{l \ldots \text{délka křivky} = \text{konst.}}$

M ... celk. hm. $M = \rho l$

$dm = \rho\, dl = \rho\sqrt{1+y'^2}\,dx$

$\vec g$

$$I[y(x)] = y_T = \frac{1}{M}\int_0^M y\,dm = \frac{1}{\cancel{\rho} l}\int_{x_1}^{x_2} y\sqrt{1+y'^2}\,\cancel{\rho}\,dx = \int_{x_1}^{x_2}\frac{y}{l}\sqrt{1+y'^2}\,dx$$

podmínka omezující variaci $l = \int_{x_1}^{x_2}\sqrt{1+y'^2}\,dx = \text{konst.}$ (tzv. Izoperimetrická podmínka)

$$\Longrightarrow \tilde I[y(x)] = I[y(x)] - \lambda\int_{x_1}^{x_2}\sqrt{1+y'^2}\,dx = \int_{x_1}^{x_2}\underbrace{\left(\frac{y}{l}-\lambda\right)\sqrt{1+y'^2}}_{F(y,y')}\,dx$$

kde $\lambda\in\mathbb{R}$ Lagr. multiplikátor

$\frac{\partial F}{\partial x}=0 \Rightarrow$ Euler. rce.

$$C_1 = F - y'\frac{\partial F}{\partial y'} = \left(\frac yl-\lambda\right)\sqrt{1+y'^2} - y'\left(\frac yl-\lambda\right)\frac{y'}{\sqrt{1+y'^2}} = \left(\frac yl - \lambda\right)\frac{1}{\sqrt{1+y'^2}}$$

$C_1^2(1+y'^2) = \left(\frac yl-\lambda\right)^2$

$y'^2 = \left(\left(\frac yl-\lambda\right)^2 - C_1^2\right)\frac{1}{C_1^2}$

$y' = \sqrt{\frac{1}{C_1^2}\left(\frac yl-\lambda\right)^2-1}$

$$1 = \frac{y'}{\sqrt{\frac{1}{C_1^2}\left(\frac yl-\lambda\right)^2-1}} \quad \Big/ \int dx$$

$$x + C_2 = \int\frac{\overbrace{y'dx}^{dy}}{\sqrt{\frac{1}{C_1^2}\left(\frac yl-\lambda\right)^2-1}} = \int\frac{C_1 l\,\cancel{\sinh z}\,dz}{\underbrace{\cancel{\sqrt{\cosh^2 z-1}}}_{\sinh z}} = \int C_1 l\,dz = C_1 l z$$

$z = \frac{x+C_2}{C_1 l}$

substituce $\frac{1}{C_1}\left(\frac yl-\lambda\right) = \cosh z \qquad \frac{dy}{C_1 l} = \sinh z\,dz$

$$\boxed{y = C_1 l\cosh\left(\frac{x+C_2}{C_1 l}\right) + \lambda l}$$

Konstanty C_1, C_2, λ určíme z $y_1 = y(x_1)$, $y_2=y(x_2)$, $l=\int_{x_1}^{x_2}\sqrt{1+y'^2}\,dx$

Izoperimetrická podmínka $\underset{\text{konst.}}{l} = \int_{x_1}^{x_2} H(x,y,y')\,dx \;\rightarrow$ def. proměnou $r(x) = \int_{x_1}^{x} H(x,y,y')\,dx$ s krajními konci $r(x_1)=0$, $r(x_2)=l$

$\Updownarrow$

$$\boxed{r' - H(x,y,y') = 0}$$ nová podmínka na r a y $\quad /\delta$

$$0 = \delta r' - \delta H = \delta r' - \frac{\partial H}{\partial y}\delta y - \frac{\partial H}{\partial y'}\delta y' \qquad / \lambda(x) \ \dots \text{Lagr. multiplikátor } \lambda=\lambda(x)$$

$$0 = \underbrace{\lambda\delta r'} - \lambda\frac{\partial H}{\partial y}\delta y - \underbrace{\lambda\frac{\partial H}{\partial y'}\delta y'}$$

$$0 = \underbrace{\frac{d}{dx}(\lambda\delta r)} - \lambda'\delta r - \lambda\frac{\partial H}{\partial y}\delta y - \left[\frac{d}{dx}\left(\lambda\frac{\partial H}{\partial y'}\delta y\right) - \frac{d}{dx}\left(\lambda\frac{\partial H}{\partial y'}\right)\delta y\right] \quad / \int_{x_1}^{x_2} dx$$

$$0 = \int_{x_1}^{x_2} -\lambda'\delta r - \left[\lambda\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\lambda\frac{\partial H}{\partial y'}\right)\right]\delta y\,dx + \underbrace{\left[\lambda\delta r\right]_{x_1}^{x_2} - \left[\lambda\frac{\partial H}{\partial y}\delta y\right]_{x_1}^{x_2}}_{\text{Pevné konce} \Rightarrow 0}$$

$$I = \int_{x_1}^{x_2} F(x,y,y')\,dx \qquad \delta I = \int_{x_1}^{x_2}\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}\right]\delta y\,dx$$

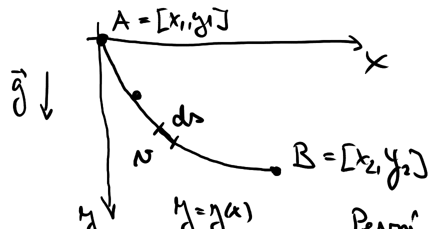
$$\tilde{I} = \int_{x_1}^{x_2} F + \underbrace{\lambda H}\,dx$$

$$0 = \int_{x_1}^{x_2}\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} + \lambda\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\lambda\frac{\partial H}{\partial y'}\right)\right]\delta y + \lambda'\delta r\,dx$$

$\lambda' = 0$, $\lambda = \text{konst.}$

$$\boxed{\begin{gathered}\frac{\partial(F+\lambda H)}{\partial y} - \frac{d}{dx}\left(\frac{\partial(F+\lambda H)}{\partial y'}\right) = 0 \\ \lambda' = 0\end{gathered}}$$

3) Najděte rovinnou křivku spojující body A, B ve svislé rovině tak, aby hmotný bod vypuštěný s nulovou počáteční rychlostí z bodu A a pohybující se po této křivce pod vlivem tíže dosáhl bodu B za nejkratší dobu.



Pevné konce

min. čas
$$T[y(x)] = \int_0^{\bar{s}} \frac{ds}{v} = \int_{x_1}^{x_2} \frac{\sqrt{1+y'^2}}{v}\,dx = \int_{x_1}^{x_2}\frac{\sqrt{1+y'^2}}{\sqrt{2gy}}\,dx = \frac{1}{\sqrt{2g}}\int_{x_1}^{x_2}\underbrace{\sqrt{\frac{1+y'^2}{y}}}_{F(y,y')}\,dx$$

$U = -mgy$

ZZE $\quad E = \frac{1}{2}mv^2 - mgy = \text{konst.} = 0$ (Brána)

$v = \sqrt{2gy}$

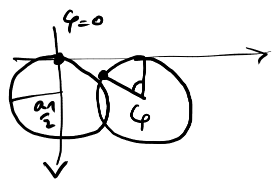
$$\frac{\partial F}{\partial x} = 0 \Rightarrow F - y'\frac{\partial F}{\partial y'} = c_1$$

$$c_1 = \sqrt{\frac{1+y'^2}{y}} - y'\frac{1}{\sqrt{y}}\frac{y'}{\sqrt{1+y'^2}} = \frac{1}{\sqrt{y}}\frac{1}{\sqrt{1+y'^2}}$$

$$y(1+y'^2) = \frac{1}{c_1^2} = a_1$$

substituce $y' = \cot\frac{\varphi}{2}$ $\quad\searrow\quad y\left(1+\frac{\cos^2\frac{\varphi}{2}}{\sin^2\frac{\varphi}{2}}\right) = y\left(\frac{1}{\sin^2\frac{\varphi}{2}}\right) = a_1 \Rightarrow y = a_1\sin^2\frac{\varphi}{2} = \underline{\underline{a_1\left(\frac{1-\cos\varphi}{2}\right)}}$

$dy = y'dx \qquad dx = \frac{1}{y'}dy = \frac{1}{\cot\frac{\varphi}{2}}a_1\frac{\sin\varphi}{2}d\varphi = \frac{\sin\frac{\varphi}{2}}{\cancel{\cos\frac{\varphi}{2}}}a_1\frac{\cancel{2}\sin\frac{\varphi}{2}\cancel{\cos\frac{\varphi}{2}}}{\cancel{2}}d\varphi = a_1\sin^2\frac{\varphi}{2}\,d\varphi = a_1\left(\frac{1-\cos\varphi}{2}\right)d\varphi \quad /\int$



$$x = \int 1\,dx = \int a_1\left(\frac{1-\cos\varphi}{2}\right)d\varphi + c_2 = a_1\left(\frac{\varphi}{2} - \frac{\sin\varphi}{2}\right) + c_2 = \underline{\underline{\frac{a_1}{2}(\varphi - \sin\varphi) + c_2}}$$