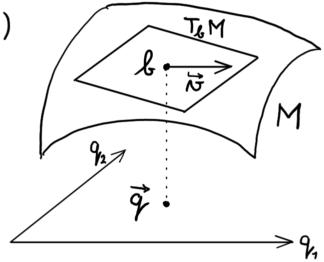
Pozorovatelné a Integrály Pohybu

Pozorovatelné a Integrály Pohybu

Pozorovatelné veličiny (polohy, rychlosti, hybnosti, momenty, síly, energie ...
jsou funkce **2s+1** proměnných $\vec{q}, \dot{\vec{q}}, t$ na tzv.
rozšířeném rychlostním fázovém prostoru

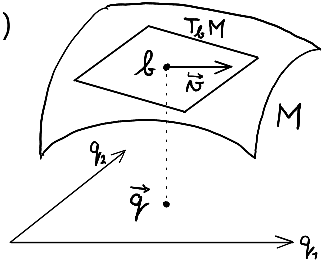
Pozorovatelné a Integrály Pohybu

Pozorovatelné veličiny (polohy, rychlosti, hybnosti, momenty, síly, energie . . .)
jsou funkce $2s+1$ proměnných $\vec{q},\dot{\vec{q}},t$ na tzv.
rozšířeném rychlostním fázovém prostoru $\underbrace{TM}_{\text{tečný bandl}}\times\mathbb{R}$ kde $TM = \bigcup_{b\in M} \underbrace{T_b M}_{\text{tečný prostor k M v bodě b}}$



Pozorovatelné a Integrály Pohybu

Pozorovatelné veličiny (polohy, rychlosti, hybnosti, momenty, síly, energie ...)
jsou funkce $2s+1$ proměnných $\vec{q}, \dot{\vec{q}}, t$ na tzv.
rozšířeném rychlostním fázovém prostoru $\underbrace{TM \times \mathbb{R}}_{\text{tečný bandl}}$ kde $TM = \bigcup_{b \in M} \underbrace{T_b M}_{\text{tečný prostor k M v bodě b}}$

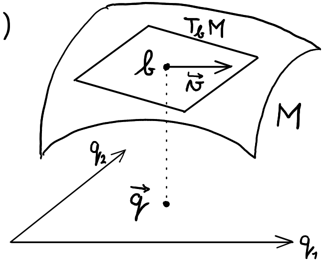


Kinetická energie v obecných souřadnicích

$$\hat{T}(\vec{q},\dot{\vec{q}},t) = T(\dot{\vec{X}}(\vec{q},\dot{\vec{q}},t)) = \frac{1}{2}\sum_{i=1}^{3N} m_i \dot{\hat{X}}_i^2 = \frac{1}{2}\sum_{i=1}^{3N} m_i \left(\frac{\partial \hat{X}_i}{\partial q_k}\dot{q}_k + \frac{\partial \hat{X}_i}{\partial t}\right)\left(\frac{\partial \hat{X}_i}{\partial q_l}\dot{q}_l + \frac{\partial \hat{X}_i}{\partial t}\right) = \frac{1}{2}\sum_{i=1}^{3N} \overbrace{m_i \frac{\partial \hat{X}_i}{\partial q_k}\frac{\partial \hat{X}_i}{\partial q_l}}^{T_{kl}(\vec{q},t)} \dot{q}_k \dot{q}_l +$$

Pozorovatelné a Integrály Pohybu

Pozorovatelné veličiny (polohy, rychlosti, hybnosti, momenty, síly, energie ...) jsou funkce $2s+1$ proměnných $\vec{q}, \dot{\vec{q}}, t$ na tzv. rozšířeném rychlostním fázovém prostoru $\underbrace{TM}_{\text{tečný bandl}} \times \mathbb{R}$ kde $TM = \bigcup_{b \in M} \underbrace{T_b M}_{\text{tečný prostor k } M \text{ v bodě } b}$



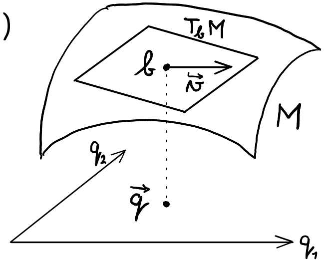
Kinetická energie v obecných souřadnicích

$$\hat{T}(\vec{q}, \dot{\vec{q}}, t) = T(\dot{\hat{\vec{X}}}(\vec{q}, \dot{\vec{q}}, t)) = \frac{1}{2} \sum_{i=1}^{3N} m_i \dot{\hat{X}}_i^2 = \frac{1}{2} \sum_{i=1}^{3N} m_i \left(\frac{\partial \hat{X}_i}{\partial q_k} \dot{q}_k + \frac{\partial \hat{X}_i}{\partial t} \right) \left(\frac{\partial \hat{X}_i}{\partial q_l} \dot{q}_l + \frac{\partial \hat{X}_i}{\partial t} \right) = \frac{1}{2} \sum_{i=1}^{3N} \overbrace{m_i \frac{\partial \hat{X}_i}{\partial q_k} \frac{\partial \hat{X}_i}{\partial q_l}}^{T_{kl}(\vec{q}, t)} \dot{q}_k \dot{q}_l +$$

$$= \frac{1}{2} \sum_{i=1}^{3N} m_i \frac{\partial \hat{X}_i}{\partial q_k} \frac{\partial \hat{X}_i}{\partial t} \dot{q}_k + \frac{1}{2} \sum_{i=1}^{3N} m_i \frac{\partial \hat{X}_i}{\partial t} \frac{\partial \hat{X}_i}{\partial q_l} \dot{q}_l + \frac{1}{2} \sum_{i=1}^{3N} m_i \frac{\partial \hat{X}_i}{\partial t} \frac{\partial \hat{X}_i}{\partial t} = \frac{1}{2} T_{kl}(\vec{q}, t) \dot{q}_k \dot{q}_l + \beta_k(\vec{q}, t) \dot{q}_k + \alpha(\vec{q}, t)$$

Pozorovatelné a Integrály Pohybu

Pozorovatelné veličiny (polohy, rychlosti, hybnosti, momenty, síly, energie ...) jsou funkce $2s+1$ proměnných $\vec{q}, \dot{\vec{q}}, t$ na tzv. rozšířeném rychlostním fázovém prostoru $\underbrace{TM}_{\text{tečný bandl}} \times \mathbb{R}$ kde $TM = \bigcup_{b \in M} \underbrace{T_b M}_{\substack{\text{tečný prostor k } M \\ \text{v bodě } b}}$



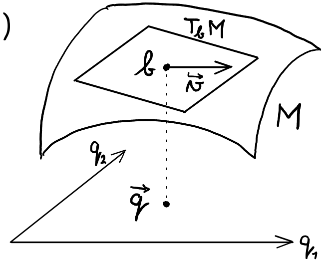
<u>Kinetická energie</u> v obecných souřadnicích

$$\hat{T}(\vec{q},\dot{\vec{q}},t) = T(\dot{\vec{\hat{X}}}(\vec{q},\dot{\vec{q}},t)) = \frac{1}{2}\sum_{i=1}^{3N} m_i \dot{\hat{X}}_i^2 = \frac{1}{2}\sum_{i=1}^{3N} m_i \left(\frac{\partial \hat{X}_i}{\partial q_k}\dot{q}_k + \frac{\partial \hat{X}_i}{\partial t}\right)\left(\frac{\partial \hat{X}_i}{\partial q_l}\dot{q}_l + \frac{\partial \hat{X}_i}{\partial t}\right) = \frac{1}{2}\sum_{i=1}^{3N} \overbrace{m_i \frac{\partial \hat{X}_i}{\partial q_k}\frac{\partial \hat{X}_i}{\partial q_l}}^{T_{kl}(\vec{q},t)} \dot{q}_k \dot{q}_l +$$

$$= \frac{1}{2}\sum_{i=1}^{3N} \underbrace{m_i \frac{\partial \hat{X}_i}{\partial q_k}\frac{\partial \hat{X}_i}{\partial t}}_{\beta_k(\vec{q},t)} \dot{q}_k + \frac{1}{2}\sum_{i=1}^{3N} \underbrace{m_i \frac{\partial \hat{X}_i}{\partial t}\frac{\partial \hat{X}_i}{\partial q_l}}_{\beta_l(\vec{q},t)} \dot{q}_l + \underbrace{\frac{1}{2}\sum_{i=1}^{3N} m_i \frac{\partial \hat{X}_i}{\partial t}\frac{\partial \hat{X}_i}{\partial t}}_{\alpha(\vec{q},t)} = \underbrace{\frac{1}{2} T_{kl}(\vec{q},t)\dot{q}_k \dot{q}_l}_{\text{P.D. kvadratická forma}} + \beta_k(\vec{q},t)\dot{q}_k + \alpha(\vec{q},t)$$

Pozorovatelné a Integrály Pohybu

Pozorovatelné veličiny (polohy, rychlosti, hybnosti, momenty, síly, energie ...)
jsou funkce $2s+1$ proměnných $\vec{q}, \dot{\vec{q}}, t$ na tzv.
rozšířeném rychlostním fázovém prostoru $\underbrace{TM}_{\text{tečný bandl}} \times \mathbb{R}$ kde $TM = \bigcup_{b \in M} \underbrace{T_b M}_{\text{tečný prostor k } M \text{ v bodě } b}$



Kinetická energie v obecných souřadnicích

$$\hat{T}(\vec{q},\dot{\vec{q}},t) = T(\dot{\hat{\vec{X}}}(\vec{q},\dot{\vec{q}},t)) = \frac{1}{2}\sum_{i=1}^{3N} m_i \dot{\hat{X}}_i^2 = \frac{1}{2}\sum_{i=1}^{3N} m_i \left(\frac{\partial \hat{X}_i}{\partial q_k}\dot{q}_k + \frac{\partial \hat{X}_i}{\partial t}\right)\left(\frac{\partial \hat{X}_i}{\partial q_l}\dot{q}_l + \frac{\partial \hat{X}_i}{\partial t}\right) = \frac{1}{2}\sum_{i=1}^{3N} \overbrace{m_i \frac{\partial \hat{X}_i}{\partial q_k}\frac{\partial \hat{X}_i}{\partial q_l}}^{T_{kl}(\vec{q},t)} \dot{q}_k \dot{q}_l +$$

$$= \frac{1}{2}\sum_{i=1}^{3N} \underbrace{m_i \frac{\partial \hat{X}_i}{\partial q_k}\frac{\partial \hat{X}_i}{\partial t}}_{\beta_k(\vec{q},t)} \dot{q}_k + \frac{1}{2}\sum_{i=1}^{3N} \underbrace{m_i \frac{\partial \hat{X}_i}{\partial t}\frac{\partial \hat{X}_i}{\partial q_l}}_{\beta_l(\vec{q},t)} \dot{q}_l + \underbrace{\frac{1}{2}\sum_{i=1}^{3N} m_i \frac{\partial \hat{X}_i}{\partial t}\frac{\partial \hat{X}_i}{\partial t}}_{\alpha(\vec{q},t)} = \underbrace{\frac{1}{2} T_{kl}(\vec{q},t)\dot{q}_k\dot{q}_l}_{\text{P.D. kvadratická forma}} + \beta_k(\vec{q},t)\dot{q}_k + \alpha(\vec{q},t)$$

- pokud jsou všechny vazby skleronomní pak $\hat{T}(\vec{q},\dot{\vec{q}}) = \frac{1}{2} T_{kl}(\vec{q})\dot{q}_k\dot{q}_l$ je homogenní stupně 2 v rychlostech

Obecná síla $\quad Q_j = \dfrac{\partial \hat{L}}{\partial q_j} + Q_j^{(nep)}$

Obecná síla $Q_j = \frac{\partial \hat{L}}{\partial q_j} + Q_j^{(nep)}$

Obecná (kanonická) hybnost $p_j = p_j(\vec{q}, \dot{\vec{q}}, t) = \frac{\partial \hat{L}}{\partial \dot{q}_j}(\vec{q}, \dot{\vec{q}}, t)$

(nemusí mít rozměr hybnosti)

Obecná síla $Q_j = \dfrac{\partial \hat{L}}{\partial q_j} + Q_j^{(nep)}$

Obecná (kanonická) hybnost $p_j = p_j(\vec{q}, \dot{\vec{q}}, t) = \dfrac{\partial \hat{L}}{\partial \dot{q}_j}(\vec{q}, \dot{\vec{q}}, t)$

(nemusí mít rozměr hybnosti)

Př. $L = \frac{1}{2} m \dot{x}_i^2 - e(\varphi - \dot{x}_i A_i)$ $\quad p_j = \dfrac{\partial L}{\partial \dot{x}_j} = m\dot{x}_j + eA_j \neq m\dot{x}_j$

Obecná síla $Q_j = \dfrac{\partial \hat{L}}{\partial q_j} + Q_j^{(nep)}$

Obecná (kanonická) hybnost $p_j = p_j(\vec{q}, \dot{\vec{q}}, t) = \dfrac{\partial \hat{L}}{\partial \dot{q}_j}(\vec{q}, \dot{\vec{q}}, t)$

(nemusí mít rozměr hybnosti)

Pozn. LR2D: $\dfrac{\hat{d}}{dt}\left(\dfrac{\partial \hat{L}}{\partial \dot{q}_j}\right) = Q_j^{(nep)} + \dfrac{\partial \hat{L}}{\partial q_j} \Leftrightarrow \dot{p}_j = Q_j$

Př. $L = \frac{1}{2} m \dot{x}_i^2 - e(\varphi - \dot{x}_i A_i)$ $\quad p_j = \dfrac{\partial L}{\partial \dot{x}_j} = m\dot{x}_j + eA_j \neq m\dot{x}_j$

Obecná síla $Q_j = \frac{\partial \hat{L}}{\partial q_j} + Q_j^{(nep)}$

Obecná (kanonická) hybnost $p_j = p_j(\vec{q}, \dot{\vec{q}}, t) = \frac{\partial \hat{L}}{\partial \dot{q}_j}(\vec{q}, \dot{\vec{q}}, t)$

(nemusí mít rozměr hybnosti)

Pozn. LR2D: $\frac{\hat{d}}{dt}\left(\frac{\partial \hat{L}}{\partial \dot{q}_j}\right) = Q_j^{(nep)} + \frac{\partial \hat{L}}{\partial q_j} \Leftrightarrow \dot{p}_j = Q_j$

Př. $L = \frac{1}{2} m \dot{x}_i^2 - e(\varphi - \dot{x}_i A_i)$ $\quad p_j = \frac{\partial L}{\partial \dot{x}_j} = m\dot{x}_j + eA_j \neq m\dot{x}_j$

Obecná energie $E = E(\vec{q}, \dot{\vec{q}}, t) = \sum_{j=1}^{s} \frac{\partial \hat{L}}{\partial \dot{q}_j} \dot{q}_j - \hat{L}$

(má rozměr energie)

Obecná síla $\quad Q_j = \dfrac{\partial \hat{L}}{\partial q_j} + Q_j^{(nep)}$

Obecná (kanonická) hybnost $\quad p_j = p_j(\vec{q}, \dot{\vec{q}}, t) = \dfrac{\partial \hat{L}}{\partial \dot{q}_j}(\vec{q}, \dot{\vec{q}}, t)$

(nemusí mít rozměr hybnosti)

Pozn. LR2D : $\dfrac{d}{dt}\left(\dfrac{\partial \hat{L}}{\partial \dot{q}_j}\right) = Q_j^{(nep)} + \dfrac{\partial \hat{L}}{\partial q_j} \Longleftrightarrow \dot{p}_j = Q_j$

Př. $L = \frac{1}{2} m \dot{x}_i^2 - e(\varphi - \dot{x}_i A_i) \qquad p_j = \dfrac{\partial L}{\partial \dot{x}_j} = m\dot{x}_j + eA_j \neq m\dot{x}_j$

Obecná energie $\quad E = E(\vec{q}, \dot{\vec{q}}, t) = \displaystyle\sum_{j=1}^{s} \frac{\partial \hat{L}}{\partial \dot{q}_j} \dot{q}_j - \hat{L}$

(má rozměr energie)

Pokud je $\hat{T}$ homogenní fce. 2. stupně v rychlostech $\dot{\vec{q}}$ (všecny vazby jsou skleronomní) a $\dfrac{\partial \hat{U}}{\partial \dot{q}_j} = 0 \ \forall j \in \hat{s}$ pak E je celková mechanická energie soustavy.

$$E = \frac{\partial \hat{L}}{\partial \dot{q}_j}\dot{q}_j - \hat{L} = \underbrace{\frac{\partial \hat{T}}{\partial \dot{q}_j}\dot{q}_j}_{=2\hat{T}} - \underbrace{\frac{\partial \hat{U}}{\partial \dot{q}_j}\dot{q}_j}_{=0} - (\hat{T} - \hat{U}) = \hat{T} + \hat{U}$$

Obecná síla $Q_j = \frac{\partial \hat{L}}{\partial q_j} + Q_j^{(nep)}$

Obecná (kanonická) hybnost $p_j = p_j(\vec{q},\dot{\vec{q}},t) = \frac{\partial \hat{L}}{\partial \dot{q}_j}(\vec{q},\dot{\vec{q}},t)$

(nemusí mít rozměr hybnosti)

Pozn. LR2D: $\frac{d}{dt}\left(\frac{\partial \hat{L}}{\partial \dot{q}_j}\right) = Q_j^{(nep)} + \frac{\partial \hat{L}}{\partial q_j} \iff \dot{p}_j = Q_j$

Př. $L = \frac{1}{2} m \dot{x}_i^2 - e(\varphi - \dot{x}_i A_i)$ $p_j = \frac{\partial L}{\partial \dot{x}_j} = m\dot{x}_j + eA_j \neq m\dot{x}_j$

Obecná energie $E = E(\vec{q},\dot{\vec{q}},t) = \sum_{j=1}^{s} \frac{\partial \hat{L}}{\partial \dot{q}_j}\dot{q}_j - \hat{L}$

(má rozměr energie)

$$E = \frac{\partial \hat{L}}{\partial \dot{q}_j}\dot{q}_j - \hat{L} = \underbrace{\frac{\partial \hat{T}}{\partial \dot{q}_j}\dot{q}_j}_{=2\hat{T}} - \underbrace{\frac{\partial \hat{U}}{\partial \dot{q}_j}\dot{q}_j}_{=0} - (\hat{T} - \hat{U}) = \hat{T} + \hat{U}$$

Pokud je $\hat{T}$ homogenní fce. 2. stupně v rychlostech $\dot{\vec{q}}$ (všecny vazby jsou skleronomní) a $\frac{\partial \hat{U}}{\partial \dot{q}_j} = 0\ \forall j \in \hat{s}$ pak E je celková mechanická energie soustavy.

Integrály pohybu (zákony zachování) – jsou 1. integrály pohyb. rovnic, fce. konstantní podél trajektorie

Obecná síla $Q_j = \frac{\partial \hat{L}}{\partial q_j} + Q_j^{(nep)}$

Obecná (kanonická) hybnost $p_j = p_j(\vec{q}, \dot{\vec{q}}, t) = \frac{\partial \hat{L}}{\partial \dot{q}_j}(\vec{q}, \dot{\vec{q}}, t)$
(nemusí mít rozměr hybnosti)

Pozn. LR2D : $\frac{\hat{d}}{dt}\left(\frac{\partial \hat{L}}{\partial \dot{q}_j}\right) = Q_j^{(nep)} + \frac{\partial \hat{L}}{\partial q_j} \iff \dot{p}_j = Q_j$

Př. $L = \frac{1}{2} m \dot{x}_i^2 - e(\varphi - \dot{x}_i A_i)$ $\quad p_j = \frac{\partial L}{\partial \dot{x}_j} = m\dot{x}_j + eA_j \neq m\dot{x}_j$

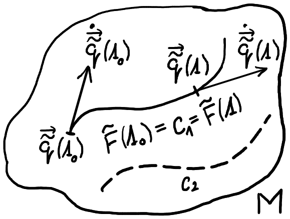
Obecná energie $E = E(\vec{q}, \dot{\vec{q}}, t) = \sum_{j=1}^{s} \frac{\partial \hat{L}}{\partial \dot{q}_j} \dot{q}_j - \hat{L}$
(má rozměr energie)

$$E = \frac{\partial \hat{L}}{\partial \dot{q}_j} \dot{q}_j - \hat{L} = \underbrace{\frac{\partial \hat{T}}{\partial \dot{q}_j} \dot{q}_j}_{=2\hat{T}} - \underbrace{\frac{\partial \hat{U}}{\partial \dot{q}_j}}_{=0} \dot{q}_j - (\hat{T} - \hat{U}) = \hat{T} + \hat{U}$$

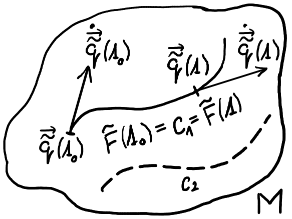
Pokud je $\hat{T}$ homogenní fce. 2. stupně v rychlostech $\dot{\vec{q}}$ (všecny vazby jsou skleronomní) a $\frac{\partial \hat{U}}{\partial \dot{q}_j} = 0 \ \forall j \in \hat{s}$ pak E je celková mechanická energie soustavy.

Integrály pohybu (zákony zachování) – jsou 1. integrály pohyb. rovnic, fce. konstantní podél trajektorie

Funkce $F(\vec{q}, \dot{\vec{q}}, t)$ je integrálem pohybu pro systém popsaný pohybovými rovnicemi $\vec{R}(\vec{q}, \dot{\vec{q}}, \ddot{\vec{q}}, t) = 0$, pokud pro každé jejich řešení $\vec{q} = \vec{q}(t)$ (tzv. trajektorii) $\exists C \in \mathbb{R}$ tak, že $\tilde{F}(t) = F(\vec{q}(t), \dot{\vec{q}}(t), t) = C \ \forall t$.

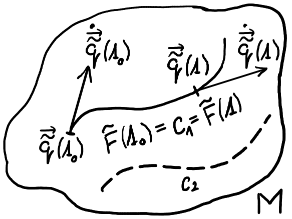

$\dot{\vec{q}}(\lambda_0)$
$\vec{q}(\lambda)$
$\dot{\vec{q}}(\lambda)$
$\vec{q}(\lambda_0)$
$\tilde{F}(\lambda_0) = c_1 = \tilde{F}(\lambda)$
c_2
M

Funkce $F = F(\vec{q}, \dot{\vec{q}}, s)$ je I. P. $\Longleftrightarrow$ $0 = \left.\frac{\hat{d}F}{ds}\right|_{\vec{R}(\vec{q}, \dot{\vec{q}}, \ddot{\vec{q}}, s) = 0} = \left.\left(\frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial F}{\partial s}\right)\right|_{\vec{R} = 0}$



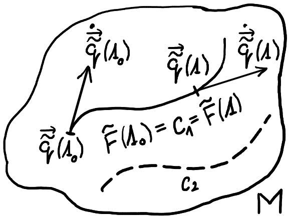
Funkce $F = F(\vec{q}, \dot{\vec{q}}, t)$ je I. P. $\Longleftrightarrow$ $$0 = \frac{\hat{d}F}{dt}\Bigg|_{\vec{R}(\vec{q},\dot{\vec{q}},\ddot{\vec{q}},t)=0} = \left(\frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial F}{\partial t}\right)\Bigg|_{\vec{R}=0}$$

$$\Rightarrow \ddot{q}_i = G_i(\vec{q}, \dot{\vec{q}}, t)$$



ověřit, že F je I. P. znamená řešit LR2D
jako lineární algebraické rovnice pro $\ddot{\vec{q}}$ a toto řešení dosadit do $\frac{\hat{d}F}{dt} = 0$

Funkce $F=F(\vec{q},\dot{\vec{q}},t)$ je I. P. $\Longleftrightarrow$ $$0=\frac{\hat{d}F}{dt}\bigg|_{\underbrace{\vec{R}(\vec{q},\dot{\vec{q}},\ddot{\vec{q}},t)=0}_{\Rightarrow\ \ddot{q}_i=G_i(\vec{q},\dot{\vec{q}},t)}}=\left(\frac{\partial F}{\partial q_i}\dot{q}_i+\frac{\partial F}{\partial \dot{q}_i}\ddot{q}_i+\frac{\partial F}{\partial t}\right)\bigg|_{\vec{R}=0}$$

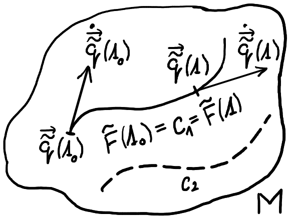


ověřit, že F je I. P. znamená řešit LR2D
jako lineární algebraické rovnice pro $\ddot{\vec{q}}$ a toto řešení dosadit do $\frac{\hat{d}F}{dt}=0$

Dále budeme předpokládat, že na soustavu nepůsobí žádné nepotenciální síly tj. $Q_j^{(nep)}=0$, pak lze některé integrály pohybu nalézt na základě chybějících proměnných v předpisu Lagrangeovy funkce:

1) čas t $\qquad \hat{L}=\hat{L}(\vec{q},\dot{\vec{q}})$ tj. $\frac{\partial\hat{L}}{\partial t}=0$ $\longrightarrow$ obecná energie $E=E(\vec{q},\dot{\vec{q}},t)=$ konst. je I. P.

Funkce $F=F(\vec{q},\dot{\vec{q}},t)$ je I. P. $\Longleftrightarrow$ $0=\frac{\hat{d}F}{dt}\Big|_{\underbrace{\vec{R}(\vec{q},\dot{\vec{q}},\ddot{\vec{q}},t)=0}_{\Rightarrow\ \ddot{q}_i=G_i(\vec{q},\dot{\vec{q}},t)}} = \left(\frac{\partial F}{\partial q_i}\dot{q}_i+\frac{\partial F}{\partial \dot{q}_i}\ddot{q}_i+\frac{\partial F}{\partial t}\right)\Big|_{\vec{R}=0}$



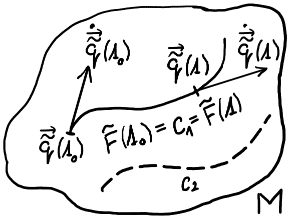
ověřit, že F je I. P. znamená řešit LR2D
jako lineární algebraické rovnice pro $\ddot{\vec{q}}$ a toto řešení dosadit do $\frac{\hat{d}F}{dt}=0$

Dále budeme předpokládat, že na soustavu nepůsobí žádné nepotenciální síly tj. $Q_j^{(nep)}=0$, pak lze některé integrály pohybu nalézt na základě chybějících proměnných v předpisu Lagrangeovy funkce:

1) čas t $\qquad \hat{L}=\hat{L}(\vec{q},\dot{\vec{q}})$ tj. $\frac{\partial\hat{L}}{\partial t}=0$ $\longrightarrow$ obecná energie $E=E(\vec{q},\dot{\vec{q}},t)=$ konst. je I. P.

$$\frac{\hat{d}E}{dt}=\frac{\hat{d}}{dt}\left(\frac{\partial\hat{L}}{\partial\dot{q}_j}\dot{q}_j-\hat{L}\right)=\frac{\hat{d}}{dt}\left(\frac{\partial\hat{L}}{\partial\dot{q}_j}\right)\dot{q}_j+\frac{\partial\hat{L}}{\partial\dot{q}_j}\ddot{q}_j-\left[\frac{\partial\hat{L}}{\partial q_j}\dot{q}_j+\frac{\partial\hat{L}}{\partial\dot{q}_j}\ddot{q}_j+\frac{\partial\hat{L}}{\partial t}\right]$$

Funkce $F=F(\vec{q},\dot{\vec{q}},t)$ je I. P. $\Leftrightarrow$ $0=\left.\frac{\hat{d}F}{dt}\right|_{\underbrace{\vec{R}(\vec{q},\dot{\vec{q}},\ddot{\vec{q}},t)=0}_{\Rightarrow\ \ddot{q}_i=G_i(\vec{q},\dot{\vec{q}},t)}} = \left.\left(\frac{\partial F}{\partial q_i}\dot{q}_i+\frac{\partial F}{\partial \dot{q}_i}\ddot{q}_i+\frac{\partial F}{\partial t}\right)\right|_{\vec{R}=0}$



ověřit, že F je I. P. znamená řešit LR2D
jako lineární algebraické rovnice pro $\ddot{\vec{q}}$ a toto řešení dosadit do $\frac{\hat{d}F}{dt}=0$

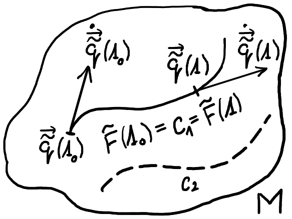
Dále budeme předpokládat, že na soustavu nepůsobí žádné nepotenciální síly tj. $Q_j^{(nep)}=0$, pak lze některé integrály pohybu nalézt na základě chybějících proměnných v předpisu Lagrangeovy funkce:

1) čas t $\quad \hat{L}=\hat{L}(\vec{q},\dot{\vec{q}})$ tj. $\frac{\partial\hat{L}}{\partial t}=0$ $\longrightarrow$ obecná energie $E=E(\vec{q},\dot{\vec{q}},t)=$ konst. je I. P.

$$\frac{\hat{d}E}{dt}=\frac{\hat{d}}{dt}\left(\frac{\partial\hat{L}}{\partial\dot{q}_j}\dot{q}_j-\hat{L}\right)=\frac{\hat{d}}{dt}\left(\frac{\partial\hat{L}}{\partial\dot{q}_j}\right)\dot{q}_j+\frac{\partial\hat{L}}{\partial\dot{q}_j}\ddot{q}_j-\left[\frac{\partial\hat{L}}{\partial q_j}\dot{q}_j+\frac{\partial\hat{L}}{\partial\dot{q}_j}\ddot{q}_j+\frac{\partial\hat{L}}{\partial t}\right]=\dot{q}_j\underbrace{\left[\frac{\hat{d}}{dt}\left(\frac{\partial\hat{L}}{\partial\dot{q}_j}\right)-\frac{\partial\hat{L}}{\partial q_j}\right]}_{\text{LR2D}\ =\ Q_j^{(nep)}}-\frac{\partial\hat{L}}{\partial t}=\underbrace{\dot{q}_jQ_j^{(nep)}}_{\text{výkon sil nepotenciálních}}-\underbrace{\frac{\partial\hat{L}}{\partial t}}_{=0}=0$$

Funkce $F=F(\vec{q},\dot{\vec{q}},t)$ je I. P. $\iff$ $0=\left.\frac{\hat{d}F}{dt}\right|_{\vec{R}(\vec{q},\dot{\vec{q}},\ddot{\vec{q}},t)=0}=\left.\left(\frac{\partial F}{\partial q_i}\dot{q}_i+\frac{\partial F}{\partial \dot{q}_i}\ddot{q}_i+\frac{\partial F}{\partial t}\right)\right|_{\vec{R}=0}$

$\Rightarrow \ddot{q}_i=G_i(\vec{q},\dot{\vec{q}},t)$



ověřit, že F je I. P. znamená řešit LR2D
jako lineární algebraické rovnice pro $\ddot{\vec{q}}$ a toto řešení dosadit do $\frac{\hat{d}F}{dt}=0$

Dále budeme předpokládat, že na soustavu nepůsobí žádné nepotenciální síly tj. $Q_j^{(nep)}=0$, pak lze některé integrály pohybu nalézt na základě chybějících proměnných v předpisu Lagrangeovy funkce:

1) čas t $\qquad \hat{L}=\hat{L}(\vec{q},\dot{\vec{q}})$ tj. $\frac{\partial \hat{L}}{\partial t}=0$ $\longrightarrow$ obecná energie $E=E(\vec{q},\dot{\vec{q}},t)=$ konst. je I. P.

$$\frac{\hat{d}E}{dt}=\frac{\hat{d}}{dt}\left(\frac{\partial \hat{L}}{\partial \dot{q}_j}\dot{q}_j-\hat{L}\right)=\frac{\hat{d}}{dt}\left(\frac{\partial \hat{L}}{\partial \dot{q}_j}\right)\dot{q}_j+\frac{\partial \hat{L}}{\partial \dot{q}_j}\ddot{q}_j-\left[\frac{\partial \hat{L}}{\partial q_j}\dot{q}_j+\frac{\partial \hat{L}}{\partial \dot{q}_j}\ddot{q}_j+\frac{\partial \hat{L}}{\partial t}\right]=\dot{q}_j\underbrace{\left[\frac{\hat{d}}{dt}\left(\frac{\partial \hat{L}}{\partial \dot{q}_j}\right)-\frac{\partial \hat{L}}{\partial q_j}\right]}_{\text{LR2D }=\,Q_j^{(nep)}}-\frac{\partial \hat{L}}{\partial t}=\underbrace{\dot{q}_jQ_j^{(nep)}}_{\text{výkon sil nepotenciálních}}-\underbrace{\frac{\partial \hat{L}}{\partial t}}_{=0}=0$$

Pozn. Obecná energie holonomní soustavy se skleronomními vazbami a konzervativními silami je konstatní.

2) **cyklická souřadnice q_k, $k \in \hat{s}$** tj. $\frac{\partial \hat{L}}{\partial q_k} = 0$

je souřadnice, na které

Lagr. fce. $\hat{L}$ nezávisí

2) **cyklická souřadnice** q_k, $k \in \hat{s}$ tj. $\frac{\partial \hat{L}}{\partial q_k} = 0$ $\longrightarrow$ obecná hybnost $p_k = p_k(\vec{q}, \dot{\vec{q}}, t) = konst.$ je I. P.

je souřadnice, na které Lagr. fce. $\hat{L}$ nezávisí

$$\frac{\hat{d}}{dt}\Big(\underbrace{\frac{\partial \hat{L}}{\partial \dot{q}_k}}_{p_k}\Big) - \underbrace{\frac{\partial \hat{L}}{\partial q_k}}_{=0} = Q_k^{(nep)} = 0 \quad \Rightarrow \quad \frac{\hat{d} p_k}{dt} = 0 \quad \text{tj.} \quad \dot{p}_k = 0$$

2) cyklická souřadnice q_k, $k \in \hat{s}$ tj. $\frac{\partial \hat{L}}{\partial q_k} = 0$ $\longrightarrow$ obecná hybnost $p_k = p_k(\vec{q}, \dot{\vec{q}}, t) = \text{konst.}$ je I. P.

je souřadnice, na které Lagr. fce. $\hat{L}$ nezávisí

$$\frac{\hat{d}}{dt}\Big(\underbrace{\frac{\partial \hat{L}}{\partial \dot{q}_k}}_{p_k}\Big) - \underbrace{\frac{\partial \hat{L}}{\partial q_k}}_{=0} = Q_k^{(nep)} = 0 \Rightarrow \frac{\hat{d} p_k}{dt} = 0 \quad \text{tj.} \quad \dot{p}_k = 0$$

Pozn. 3) $\frac{\partial \hat{L}}{\partial \dot{q}_k} = 0 \Rightarrow \frac{\partial \hat{L}}{\partial q_k} = Q_k^{(nep)} = 0$ nenastává

2) cyklická souřadnice q_k, $k \in \hat{s}$ tj. $\frac{\partial \hat{L}}{\partial q_k} = 0$ $\longrightarrow$ obeacná hybnost $p_k = p_k(\vec{q}, \dot{\vec{q}}, t) = \text{konst.}$ je I. P.

je souřadnice, na které Lagr. fce. $\hat{L}$ nezávisí

$$\frac{\hat{d}}{dt}\Bigg(\underbrace{\frac{\partial \hat{L}}{\partial \dot{q}_k}}_{p_k}\Bigg) - \underbrace{\frac{\partial \hat{L}}{\partial q_k}}_{=0} = Q_k^{(nep)} = 0 \quad \Rightarrow \quad \frac{\hat{d} p_k}{dt} = 0 \quad \text{tj.} \quad \dot{p}_k = 0$$

Pozn. 3) $\frac{\partial \hat{L}}{\partial \dot{q}_k} = 0 \Rightarrow \frac{\partial \hat{L}}{\partial q_k} = Q_k^{(nep)} = 0$ nenastává

Teorém Noetherové (1915) "Ke každé jednoparametrické grupě tranosformací konfiguračního prostoru které ponechávají Lagrangeovu funkci invariantní (symetrie Lagr. fce.) existuje integrál pohybu."

2) cyklická souřadnice q_k, $k \in \hat{s}$ tj. $\frac{\partial \hat{L}}{\partial q_k} = 0$ $\longrightarrow$ obeacná hybnost $p_k = p_k(\vec{q}, \dot{\vec{q}}, t) = \text{konst.}$ je I. P.

je souřadnice, na které Lagr. fce. $\hat{L}$ nezávisí

$$\frac{\hat{d}}{dt}\underbrace{\left(\frac{\partial \hat{L}}{\partial \dot{q}_k}\right)}_{p_k} - \underbrace{\frac{\partial \hat{L}}{\partial q_k}}_{=0} = Q_k^{(nep)} = 0 \Rightarrow \frac{\hat{d} p_k}{dt} = 0 \quad \text{tj.} \quad \dot{p}_k = 0$$

Pozn. 3) $\frac{\partial \hat{L}}{\partial \dot{q}_k} = 0 \Rightarrow \frac{\partial \hat{L}}{\partial q_k} = Q_k^{(nep)} = 0$ nenastává

Teorém Noetherové (1915) "Ke každé jednoparametrické grupě tranosformací konfiguračního prostoru které ponechávají Lagrangeovu funkci invariantní (symetrie Lagr. fce.) existuje integrál pohybu."

Jednoparametrická grupa transformací M je spojitý homomorfizmus grup $\phi : (\mathbb{R}, +) \to (\mathrm{Dif}(M), \circ)$

$$\phi : \varepsilon \to \phi^{\varepsilon} \qquad \phi^0 = \mathrm{Id} \qquad \phi^{\varepsilon+\delta} = \phi^{\varepsilon} \circ \phi^{\delta} \qquad (\phi^{\varepsilon})^{-1} = \phi^{-\varepsilon}$$

2) cyklická souřadnice $q_k,\ k\in\hat{s}$ tj. $\frac{\partial\hat{L}}{\partial q_k}=0 \longrightarrow$ obecná hybnost $p_k = p_k(\vec{q},\dot{\vec{q}},t) = \text{konst.}$ je I. P.

je souřadnice, na které Lagr. fce. $\hat{L}$ nezávisí

$$\frac{\hat{d}}{dt}\Big(\underbrace{\frac{\partial\hat{L}}{\partial\dot{q}_k}}_{p_k}\Big) - \underbrace{\frac{\partial\hat{L}}{\partial q_k}}_{=0} = Q_k^{(nep)} = 0 \quad\Rightarrow\quad \frac{\hat{d}p_k}{dt}=0 \quad \text{tj.} \quad \dot{p}_k=0$$

Pozn. 3) $\frac{\partial\hat{L}}{\partial\dot{q}_k}=0 \Rightarrow \frac{\partial\hat{L}}{\partial q_k} = Q_k^{(nep)} = 0$ nenastává

Teorém Noetherové (1915) "Ke každé jednoparametrické grupě tranosformací konfiguračního prostoru které ponechávají Lagrangeovu funkci invariantní (symetrie Lagr. fce.) existuje integrál pohybu."

Jednoparametrická grupa transformací M je spojitý homomorfizmus grup $\phi:(\mathbb{R},+)\to(\mathrm{Dif}(M),\circ)$

$$\phi:\varepsilon\to\phi^\varepsilon \qquad \phi^0=\mathrm{Id} \qquad \phi^{\varepsilon+\delta}=\phi^\varepsilon\circ\phi^\delta \qquad (\phi^\varepsilon)^{-1}=\phi^{-\varepsilon}$$

Transformace (aktivní) $\phi^\varepsilon\in\mathrm{Dif}(M)$ je zde bijekce $\phi^\varepsilon:M\to M$ taková, že ϕ^ε a $(\phi^\varepsilon)^{-1}$ jsou třídy $C^{(1)}$.

Znění teorému Noetherové, které dokážeme:

Transformace (aktivní)

$q'_j = q'_j(\vec{q}, \varepsilon) = \phi^{\varepsilon}_j(\vec{q})$ fce. třídy $C^{(2)}$, $\det\left(\frac{\partial q'_i}{\partial q_j}\right) \neq 0$

$q'_j(\vec{q}, 0) = q_j \qquad \forall j \in \hat{s}$

Znění teorému Noetherové, které dokážeme:

Transformace (aktivní)

$$q'_j = q'_j(\vec{q},\varepsilon) = \phi^{\varepsilon}_j(\vec{q}) \quad \text{fce. třídy } C^{(2)},\ \det\left(\frac{\partial q'_i}{\partial q_j}\right) \neq 0$$

$$q'_j(\vec{q},0) = q_j \qquad \forall j \in \hat{s}$$

$$\dot{q}'_j = \dot{q}'_j(\vec{q},\dot{\vec{q}},\varepsilon) = \phi^{\varepsilon}_{*j}(\vec{q},\dot{\vec{q}}) = \frac{\hat{d} q'_j}{dt} = \frac{\partial q'_j}{\partial q_k}\dot{q}_k$$

Znění teorému Noetherové, které dokážeme:

Transformace (aktivní)

$$q'_j = q'_j(\vec{q},\varepsilon) = \phi^{\varepsilon}_j(\vec{q}) \quad \text{fce. třídy } C^{(2)},\ \det\left(\frac{\partial q'_i}{\partial q_j}\right) \neq 0$$

$$q'_j(\vec{q},0) = q_j \qquad \forall j \in \hat{s}$$

$$\dot{q}'_j = \dot{q}'_j(\vec{q},\dot{\vec{q}},\varepsilon) = \phi^{\varepsilon}_{*j}(\vec{q},\dot{\vec{q}}) = \frac{\hat{d}\, q'_j}{dt} = \frac{\partial q'_j}{\partial q_k}\dot{q}_k$$

Invariance Lagrangeovy funkce $\forall \vec{q}\ \forall \dot{\vec{q}}\ \forall t\ \forall \varepsilon$

$$L'(\vec{q},\dot{\vec{q}},t,\varepsilon) := L(\vec{q}'(\vec{q},\varepsilon), \dot{\vec{q}}'(\vec{q},\dot{\vec{q}},\varepsilon), t) = L(\vec{q},\dot{\vec{q}},t)$$

Znění teorému Noetherové, které dokážeme:

Transformace (aktivní)

$$q_j' = q_j'(\vec{q},\varepsilon) = \phi_j^{\varepsilon}(\vec{q}) \quad \text{fce. třídy } C^{(2)},\ \det\left(\frac{\partial q_i'}{\partial q_j}\right) \neq 0$$

$$q_j'(\vec{q},0) = q_j \qquad \forall j \in \hat{s}$$

$$\dot{q}_j' = \dot{q}_j'(\vec{q},\dot{\vec{q}},\varepsilon) = \phi^{\varepsilon}_{*j}(\vec{q},\dot{\vec{q}}) = \frac{\hat{d}\, q_j'}{dt} = \frac{\partial q_j'}{\partial q_k}\dot{q}_k$$

Invariance Lagrangeovy funkce $\forall \vec{q}\ \forall \dot{\vec{q}}\ \forall t\ \forall \varepsilon$

$$L'(\vec{q},\dot{\vec{q}},t,\varepsilon) := L(\vec{q}\,'(\vec{q},\varepsilon), \dot{\vec{q}}\,'(\vec{q},\dot{\vec{q}},\varepsilon), t) = L(\vec{q},\dot{\vec{q}},t)$$

$$\Downarrow$$

Veličina $I(\vec{q},\dot{\vec{q}},t) = \sum_{k=1}^{s} \frac{\partial L}{\partial \dot{q}_k}\left(\frac{\partial q_k'}{\partial \varepsilon}\right)_{\varepsilon=0}$ je I. P.

Znění teorému Noetherové, které dokážeme:

Transformace (aktivní)

$$q_j' = q_j'(\vec{q},\varepsilon) = \phi_j^{\varepsilon}(\vec{q}) \quad \text{fce. třídy } C^{(2)},\ \det\left(\frac{\partial q_i'}{\partial q_j}\right) \neq 0$$

$$q_j'(\vec{q},0) = q_j \qquad \forall j \in \hat{s}$$

$$\dot{q}_j' = \dot{q}_j'(\vec{q},\dot{\vec{q}},\varepsilon) = \phi^{\varepsilon}_{*j}(\vec{q},\dot{\vec{q}}) = \frac{\hat{d}\, q_j'}{dt} = \frac{\partial q_j'}{\partial q_k}\dot{q}_k$$

Invariance Lagrangeovy funkce $\forall \vec{q}\ \forall \dot{\vec{q}}\ \forall t\ \forall \varepsilon$

$$L'(\vec{q},\dot{\vec{q}},t,\varepsilon) := L(\vec{q}'(\vec{q},\varepsilon), \dot{\vec{q}}'(\vec{q},\dot{\vec{q}},\varepsilon), t) = L(\vec{q},\dot{\vec{q}},t)$$

$$\Downarrow$$

Veličina $I(\vec{q},\dot{\vec{q}},t) = \sum_{k=1}^{s} \frac{\partial L}{\partial \dot{q}_k}\left(\frac{\partial q_k'}{\partial \varepsilon}\right)_{\varepsilon=0}$ je I. P.

Důkaz: invariance $L \iff \forall \varepsilon \forall \vec{q} \forall \dot{\vec{q}} \forall t$

$$0 = \frac{\partial L'}{\partial \varepsilon}\bigg|_{(\vec{q},\dot{\vec{q}},t,\varepsilon)} = \frac{\partial L}{\partial q_i}\bigg|_{(\vec{q}',\dot{\vec{q}}',t)} \cdot \frac{\partial q_i'}{\partial \varepsilon}\bigg|_{(\vec{q},\varepsilon)} + \frac{\partial L}{\partial \dot{q}_i}\bigg|_{(\vec{q}',\dot{\vec{q}}',t)} \cdot \frac{\partial \dot{q}_i'}{\partial \varepsilon}\bigg|_{(\vec{q},\dot{\vec{q}},\varepsilon)} =$$

Znění teorému Noetherové, které dokážeme:

Transformace (aktivní)

$$q'_j = q'_j(\vec{q},\varepsilon) = \phi^{\varepsilon}_j(\vec{q}) \quad \text{fce. třídy } C^{(2)},\ \det\left(\frac{\partial q'_i}{\partial q_j}\right) \neq 0$$

$$q'_j(\vec{q},0) = q_j \qquad \forall j \in \hat{s}$$

$$\dot{q}'_j = \dot{q}'_j(\vec{q},\dot{\vec{q}},\varepsilon) = \phi^{\varepsilon}_{*j}(\vec{q},\dot{\vec{q}}) = \frac{\hat{d}\, q'_j}{dt} = \frac{\partial q'_j}{\partial q_k}\dot{q}_k$$

Invariance Lagrangeovy funkce $\forall \vec{q}\ \forall \dot{\vec{q}}\ \forall t\ \forall \varepsilon$

$$L'(\vec{q},\dot{\vec{q}},t,\varepsilon) := L(\vec{q}'(\vec{q},\varepsilon),\dot{\vec{q}}'(\vec{q},\dot{\vec{q}},\varepsilon),t) = L(\vec{q},\dot{\vec{q}},t)$$

$$\Downarrow$$

Veličina $I(\vec{q},\dot{\vec{q}},t) = \sum_{k=1}^{s} \frac{\partial L}{\partial \dot{q}_k}\left(\frac{\partial q'_k}{\partial \varepsilon}\right)_{\varepsilon=0}$ je I. P.

Důkaz: invariance $L \iff \forall\varepsilon \forall\vec{q} \forall\dot{\vec{q}} \forall t$

$$0 = \left.\frac{\partial L'}{\partial \varepsilon}\right|_{(\vec{q},\dot{\vec{q}},t,\varepsilon)} = \left.\frac{\partial L}{\partial q_i}\right|_{(\vec{q}',\dot{\vec{q}}',t)} \cdot \left.\frac{\partial q'_i}{\partial \varepsilon}\right|_{(\vec{q},\varepsilon)} + \left.\frac{\partial L}{\partial \dot{q}_i}\right|_{(\vec{q}',\dot{\vec{q}}',t)} \cdot \left.\frac{\partial \dot{q}'_i}{\partial \varepsilon}\right|_{(\vec{q},\dot{\vec{q}},\varepsilon)} =$$

$$\frac{\partial \dot{q}'_i}{\partial \varepsilon} = \frac{\partial}{\partial \varepsilon}\left(\frac{\hat{d} q'_i}{dt}\right) = \frac{\partial}{\partial \varepsilon}\left(\frac{\partial q'_i}{\partial q_k}\dot{q}_k\right) = \frac{\partial}{\partial \varepsilon}\left(\frac{\partial q'_i}{\partial q_k}\right)\dot{q}_k = \frac{\partial}{\partial q_k}\left(\frac{\partial q'_i}{\partial \varepsilon}\right)\dot{q}_k = \frac{\hat{d}}{dt}\left(\frac{\partial q'_i}{\partial \varepsilon}\right)$$

Znění teorému Noetherové, které dokážeme:

Transformace (aktivní)

$$q'_j = q'_j(\vec{q},\varepsilon) = \phi^{\varepsilon}_j(\vec{q}) \quad \text{fce. třídy } C^{(2)},\ \det\left(\frac{\partial q'_i}{\partial q_j}\right) \neq 0$$

$$q'_j(\vec{q},0) = q_j \qquad \forall j \in \hat{s}$$

$$\dot{q}'_j = \dot{q}'_j(\vec{q},\dot{\vec{q}},\varepsilon) = \phi^{\varepsilon}_{*j}(\vec{q},\dot{\vec{q}}) = \frac{\hat{d} q'_j}{dt} = \frac{\partial q'_j}{\partial q_k}\dot{q}_k$$

Invariance Lagrangeovy funkce $\forall\vec{q}\ \forall\dot{\vec{q}}\ \forall t\ \forall\varepsilon$

$$L'(\vec{q},\dot{\vec{q}},t,\varepsilon) := L(\vec{q}'(\vec{q},\varepsilon), \dot{\vec{q}}'(\vec{q},\dot{\vec{q}},\varepsilon), t) = L(\vec{q},\dot{\vec{q}},t)$$

$$\Downarrow$$

Veličina $I(\vec{q},\dot{\vec{q}},t) = \sum_{k=1}^{\hat{s}} \frac{\partial L}{\partial \dot{q}_k}\left(\frac{\partial q'_k}{\partial \varepsilon}\right)_{\varepsilon=0}$ je I. P.

Důkaz: invariance $L \iff \forall\varepsilon\forall\vec{q}\forall\dot{\vec{q}}\forall t$

$$0 = \frac{\partial L'}{\partial\varepsilon}\bigg|_{(\vec{q},\dot{\vec{q}},t,\varepsilon)} = \frac{\partial L}{\partial q_i}\bigg|_{(\vec{q}',\dot{\vec{q}}',t)} \cdot \frac{\partial q'_i}{\partial\varepsilon}\bigg|_{(\vec{q},\varepsilon)} + \frac{\partial L}{\partial \dot{q}_i}\bigg|_{(\vec{q}',\dot{\vec{q}}',t)} \cdot \frac{\partial \dot{q}'_i}{\partial\varepsilon}\bigg|_{(\vec{q},\dot{\vec{q}},\varepsilon)} = \frac{\partial L}{\partial q_i}\frac{\partial q'_i}{\partial\varepsilon} + \frac{\partial L}{\partial \dot{q}_i}\frac{\hat{d}}{dt}\left(\frac{\partial q'_i}{\partial\varepsilon}\right) = \frac{\partial L}{\partial q_i}\frac{\partial q'_i}{\partial\varepsilon} + \frac{\hat{d}}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\frac{\partial q'_i}{\partial\varepsilon}\right) - \frac{\hat{d}}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\frac{\partial q'_i}{\partial\varepsilon} =$$

$$\frac{\partial \dot{q}'_i}{\partial\varepsilon} = \frac{\partial}{\partial\varepsilon}\left(\frac{\hat{d} q'_i}{dt}\right) = \frac{\partial}{\partial\varepsilon}\left(\frac{\partial q'_i}{\partial q_k}\dot{q}_k\right) = \frac{\partial}{\partial\varepsilon}\left(\frac{\partial q'_i}{\partial q_k}\right)\dot{q}_k = \frac{\partial}{\partial q_k}\left(\frac{\partial q'_i}{\partial\varepsilon}\right)\dot{q}_k = \frac{\hat{d}}{dt}\left(\frac{\partial q'_i}{\partial\varepsilon}\right)$$

Znění teorému Noetherové, které dokážeme:

Transformace (aktivní)

$$q'_j = q'_j(\vec{q},\varepsilon) = \phi^\varepsilon_j(\vec{q}) \quad \text{fce. třídy } C^{(2)},\ \det\left(\frac{\partial q'_i}{\partial q_j}\right) \neq 0$$

$$q'_j(\vec{q},0) = q_j \qquad \forall j \in \hat{s}$$

$$\dot{q}'_j = \dot{q}'_j(\vec{q},\dot{\vec{q}},\varepsilon) = \phi^\varepsilon_{*j}(\vec{q},\dot{\vec{q}}) = \frac{\hat{d}\, q'_j}{d t} = \frac{\partial q'_j}{\partial q_k}\dot{q}_k$$

Invariance Lagrangeovy funkce $\forall \vec{q}\ \forall \dot{\vec{q}}\ \forall t\ \forall \varepsilon$

$$L'(\vec{q},\dot{\vec{q}},t,\varepsilon) := L(\vec{q}'(\vec{q},\varepsilon), \dot{\vec{q}}'(\vec{q},\dot{\vec{q}},\varepsilon), t) = L(\vec{q},\dot{\vec{q}},t)$$

$$\Downarrow$$

Veličina $I(\vec{q},\dot{\vec{q}},t) = \sum_{k=1}^{\hat{s}} \frac{\partial L}{\partial \dot{q}_k}\left(\frac{\partial q'_k}{\partial \varepsilon}\right)_{\varepsilon=0}$ je I. P.

Důkaz: invariance $L \iff \forall \varepsilon \forall \vec{q} \forall \dot{\vec{q}} \forall t$

$$0 = \frac{\partial L'}{\partial \varepsilon}\bigg|_{(\vec{q},\dot{\vec{q}},t,\varepsilon)} = \frac{\partial L}{\partial q_i}\bigg|_{(\vec{q}',\dot{\vec{q}}',t)} \cdot \frac{\partial q'_i}{\partial \varepsilon}\bigg|_{(\vec{q},\varepsilon)} + \frac{\partial L}{\partial \dot{q}_i}\bigg|_{(\vec{q}',\dot{\vec{q}}',t)} \cdot \frac{\partial \dot{q}'_i}{\partial \varepsilon}\bigg|_{(\vec{q},\dot{\vec{q}},\varepsilon)} = \frac{\partial L}{\partial q_i}\frac{\partial q'_i}{\partial \varepsilon} + \frac{\partial L}{\partial \dot{q}_i}\frac{\hat{d}}{dt}\left(\frac{\partial q'_i}{\partial \varepsilon}\right) = \frac{\partial L}{\partial q_i}\frac{\partial q'_i}{\partial \varepsilon} + \frac{\hat{d}}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\frac{\partial q'_i}{\partial \varepsilon}\right) - \frac{\hat{d}}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\frac{\partial q'_i}{\partial \varepsilon} =$$

$$\frac{\partial \dot{q}'_i}{\partial \varepsilon} = \frac{\partial}{\partial \varepsilon}\left(\frac{\hat{d} q'_i}{dt}\right) = \frac{\partial}{\partial \varepsilon}\left(\frac{\partial q'_i}{\partial q_k}\dot{q}_k\right) = \frac{\partial}{\partial \varepsilon}\left(\frac{\partial q'_i}{\partial q_k}\right)\dot{q}_k = \frac{\partial}{\partial q_k}\left(\frac{\partial q'_i}{\partial \varepsilon}\right)\dot{q}_k = \frac{\hat{d}}{dt}\left(\frac{\partial q'_i}{\partial \varepsilon}\right)$$

$$= \left[\frac{\partial L}{\partial q_i} - \frac{\hat{d}}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right]\bigg|_{(\vec{q}',\dot{\vec{q}}',t)} \cdot \frac{\partial q'_i}{\partial \varepsilon} + \frac{\hat{d}}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\frac{\partial q'_i}{\partial \varepsilon}\right)$$

Znění teorému Noetherové, které dokážeme:

Transformace (aktivní)

$$q'_j = q'_j(\vec{q},\varepsilon) = \phi^{\varepsilon}_j(\vec{q}) \quad \text{fce. třídy } C^{(2)},\ \det\left(\frac{\partial q'_i}{\partial q_j}\right) \neq 0$$

$$q'_j(\vec{q},0) = q_j \qquad \forall j \in \hat{s}$$

$$\dot{q}'_j = \dot{q}'_j(\vec{q},\dot{\vec{q}},\varepsilon) = \phi^{\varepsilon}_{*j}(\vec{q},\dot{\vec{q}}) = \frac{\hat{d}\, q'_j}{dt} = \frac{\partial q'_j}{\partial q_k}\dot{q}_k$$

Invariance Lagrangeovy funkce $\forall \vec{q}\ \forall \dot{\vec{q}}\ \forall t\ \forall \varepsilon$

$$L'(\vec{q},\dot{\vec{q}},t,\varepsilon) := L(\vec{q}'(\vec{q},\varepsilon), \dot{\vec{q}}'(\vec{q},\dot{\vec{q}},\varepsilon), t) = L(\vec{q},\dot{\vec{q}},t)$$

$$\Downarrow$$

Veličina $I(\vec{q},\dot{\vec{q}},t) = \sum_{k=1}^{s} \frac{\partial L}{\partial \dot{q}_k}\left(\frac{\partial q'_k}{\partial \varepsilon}\right)_{\varepsilon=0}$ je I. P.

Důkaz: invariance $L \iff \forall\varepsilon\,\forall\vec{q}\,\forall\dot{\vec{q}}\,\forall t$

$$0 = \frac{\partial L'}{\partial \varepsilon}\bigg|_{(\vec{q},\dot{\vec{q}},t,\varepsilon)} = \frac{\partial L}{\partial q_i}\bigg|_{(\vec{q}',\dot{\vec{q}}',t)} \cdot \frac{\partial q'_i}{\partial \varepsilon}\bigg|_{(\vec{q},\varepsilon)} + \frac{\partial L}{\partial \dot{q}_i}\bigg|_{(\vec{q}',\dot{\vec{q}}',t)} \cdot \frac{\partial \dot{q}'_i}{\partial \varepsilon}\bigg|_{(\vec{q},\dot{\vec{q}},\varepsilon)} = \frac{\partial L}{\partial q_i}\frac{\partial q'_i}{\partial \varepsilon} + \frac{\partial L}{\partial \dot{q}_i}\frac{\hat{d}}{dt}\left(\frac{\partial q'_i}{\partial \varepsilon}\right) = \frac{\partial L}{\partial q_i}\frac{\partial q'_i}{\partial \varepsilon} + \frac{\hat{d}}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\frac{\partial q'_i}{\partial \varepsilon}\right) - \frac{\hat{d}}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\frac{\partial q'_i}{\partial \varepsilon} =$$

$$\frac{\partial \dot{q}'_i}{\partial \varepsilon} = \frac{\partial}{\partial \varepsilon}\left(\frac{\hat{d} q'_i}{dt}\right) = \frac{\partial}{\partial \varepsilon}\left(\frac{\partial q'_i}{\partial q_k}\dot{q}_k\right) = \frac{\partial}{\partial \varepsilon}\left(\frac{\partial q'_i}{\partial q_k}\right)\dot{q}_k = \frac{\partial}{\partial q_k}\left(\frac{\partial q'_i}{\partial \varepsilon}\right)\dot{q}_k = \frac{\hat{d}}{dt}\left(\frac{\partial q'_i}{\partial \varepsilon}\right)$$

$$= \left[\frac{\partial L}{\partial q_i} - \frac{\hat{d}}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right]\bigg|_{(\vec{q}',\dot{\vec{q}}',t)} \cdot \frac{\partial q'_i}{\partial \varepsilon} + \frac{\hat{d}}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\frac{\partial q'_i}{\partial \varepsilon}\right)$$

zeslabíme požadavky z $\forall\varepsilon$ na $\varepsilon = 0$ a dosadíme z LR2D

$$0 = \frac{\partial L'}{\partial \varepsilon}\bigg|_{\varepsilon=0} = \underbrace{\left[\frac{\partial L}{\partial q_i} - \frac{\hat{d}}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right]\bigg|_{(\vec{q},\dot{\vec{q}},t)}}_{= -Q_i^{(nep)} = 0\ \forall i \in \hat{s}} \cdot \frac{\partial q'_i}{\partial \varepsilon}\bigg|_{\varepsilon=0} + \frac{\hat{d}}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\frac{\partial q'_i}{\partial \varepsilon}\bigg|_{\varepsilon=0}\right)\bigg|_{\vec{R}=0} = \frac{\hat{d}}{dt}(I)\bigg|_{\vec{R}=0}$$

LR2D $\vec{R}(\vec{q},\dot{\vec{q}},\ddot{\vec{q}},t) = 0 \Rightarrow$

QED.

Pozn. $L'(\vec{q},\dot{\vec{q}},t,\varepsilon) = L'(\vec{q},\dot{\vec{q}},t,0) + \left.\frac{\partial L'}{\partial \varepsilon}\right|_{\varepsilon=0} \cdot \varepsilon + \mathcal{O}(\varepsilon^2)$

$0 = \left.\frac{\partial L'}{\partial \varepsilon}\right|_{\varepsilon=0} \cdot \varepsilon = \text{lin. část}\left[L'(\vec{q},\dot{\vec{q}},t,\varepsilon) - L(\vec{q},\dot{\vec{q}},t)\right]$

Pozn. $L'(\vec{q},\dot{\vec{q}},t,\varepsilon) = L'(\vec{q},\dot{\vec{q}},t,0) + \left.\frac{\partial L'}{\partial \varepsilon}\right|_{\varepsilon=0} \cdot \varepsilon + \mathcal{O}(\varepsilon^2)$ $\qquad 0 = \left.\frac{\partial L'}{\partial \varepsilon}\right|_{\varepsilon=0} \cdot \varepsilon = \text{lin. část}\left[L'(\vec{q},\dot{\vec{q}},t,\varepsilon) - L(\vec{q},\dot{\vec{q}},t)\right]$

Infinitesimální verze teorému Noetherové:

Transfromace (Taylorův rozvoj do 1. řádu v ε)

$$q_j' = q_j'(\vec{q},0) + \left.\frac{\partial q_j'}{\partial \varepsilon}\right|_{\varepsilon=0} \cdot \varepsilon + \mathcal{O}(\varepsilon^2) = q_j + Y_j \cdot \varepsilon \qquad \underbrace{Y_j = \left.\frac{\partial q_j'}{\partial \varepsilon}\right|_{\varepsilon=0} = Y_j(\vec{q})}_{\text{vektorové pole tzv. generátor transformace}}$$

Pozn. $L'(\vec{q},\dot{\vec{q}},t,\varepsilon) = L'(\vec{q},\dot{\vec{q}},t,0) + \left.\frac{\partial L'}{\partial \varepsilon}\right|_{\varepsilon=0} \cdot \varepsilon + O(\varepsilon^2)$ $\qquad 0 = \left.\frac{\partial L'}{\partial \varepsilon}\right|_{\varepsilon=0} \cdot \varepsilon = \text{lin. část}\left[L'(\vec{q},\dot{\vec{q}},t,\varepsilon) - L(\vec{q},\dot{\vec{q}},t) \right]$

Infinitesimální verze teorému Noetherové:

Transfromace (Taylorův rozvoj do 1. řádu v ε)

$$q_j' = q_j'(\vec{q},0) + \left.\frac{\partial q_j'}{\partial \varepsilon}\right|_{\varepsilon=0} \cdot \varepsilon + O(\varepsilon^2) = q_j + Y_j \cdot \varepsilon \qquad \underbrace{Y_j = \left.\frac{\partial q_j'}{\partial \varepsilon}\right|_{\varepsilon=0} = Y_j(\vec{q})}_{\text{vektorové pole tzv. generátor transformace}}$$

$$\dot{q}_j' = \frac{\hat{d} q_j'}{dt} = \dot{q}_j + \dot{Y}_j \cdot \varepsilon \quad \text{kde } \dot{Y}_j = \frac{\partial Y_j}{\partial q_i} \dot{q}_i$$

Pozn. $L'(\vec{q},\dot{\vec{q}},t,\varepsilon) = L'(\vec{q},\dot{\vec{q}},t,0) + \left.\frac{\partial L'}{\partial \varepsilon}\right|_{\varepsilon=0} \cdot \varepsilon + \mathcal{O}(\varepsilon^2)$ $\qquad 0 = \left.\frac{\partial L'}{\partial \varepsilon}\right|_{\varepsilon=0} \cdot \varepsilon = \text{lin. část}\left[L'(\vec{q},\dot{\vec{q}},t,\varepsilon) - L(\vec{q},\dot{\vec{q}},t)\right]$

Infinitesimální verze teorému Noetherové:

Transfromace (Taylorův rozvoj do 1. řádu v ε)

$$q'_j = q'_j(\vec{q},0) + \left.\frac{\partial q'_j}{\partial \varepsilon}\right|_{\varepsilon=0} \cdot \varepsilon + \mathcal{O}(\varepsilon^2) = q_j + Y_j \cdot \varepsilon \qquad \underbrace{Y_j = \left.\frac{\partial q'_j}{\partial \varepsilon}\right|_{\varepsilon=0} = Y_j(\vec{q})}_{\text{vektorové pole tzv. generátor transformace}}$$

$$\dot{q}'_j = \frac{\hat{d} q'_j}{dt} = \dot{q}_j + \dot{Y}_j \cdot \varepsilon \quad \text{kde} \quad \dot{Y}_j = \frac{\partial Y_j}{\partial q_i} \dot{q}_i$$

Invariance L do 1. řádu v ε $\quad \forall \vec{q}\ \forall \dot{\vec{q}}\ \forall t$

$$\left.\frac{\partial L'}{\partial \varepsilon}\right|_{\varepsilon=0} = \frac{\partial L}{\partial q_i} Y_i + \frac{\partial L}{\partial \dot{q}_i} \dot{Y}_i = 0$$

Pozn. $L'(\vec{q},\dot{\vec{q}},t,\varepsilon) = L'(\vec{q},\dot{\vec{q}},t,0) + \frac{\partial L'}{\partial \varepsilon}\Big|_{\varepsilon=0}\cdot\varepsilon + O(\varepsilon^2)$ $\qquad$ $0 = \frac{\partial L'}{\partial \varepsilon}\Big|_{\varepsilon=0}\cdot\varepsilon = \text{lin. část}\left[L'(\vec{q},\dot{\vec{q}},t,\varepsilon) - L(\vec{q},\dot{\vec{q}},t)\right]$

Infinitesimální verze teorému Noetherové:

Transfromace (Taylorův rozvoj do 1. řádu v ε)

$$q'_j = q'_j(\vec{q},0) + \frac{\partial q'_j}{\partial \varepsilon}\Big|_{\varepsilon=0}\cdot\varepsilon + O(\varepsilon^2) = q_j + Y_j\cdot\varepsilon \qquad \underbrace{Y_j = \frac{\partial q'_j}{\partial \varepsilon}\Big|_{\varepsilon=0} = Y_j(\vec{q})}_{\text{vektorové pole tzv. generátor transformace}}$$

$$\dot{q}'_j = \frac{\hat{d} q'_j}{dt} = \dot{q}_j + \dot{Y}_j\cdot\varepsilon \quad \text{kde} \quad \dot{Y}_j = \frac{\partial Y_j}{\partial q_i}\dot{q}_i$$

Invariance L do 1. řádu v ε $\quad \forall \vec{q}\ \forall \dot{\vec{q}}\ \forall t$

$$\frac{\partial L'}{\partial \varepsilon}\Big|_{\varepsilon=0} = \frac{\partial L}{\partial q_i} Y_i + \frac{\partial L}{\partial \dot{q}_i}\dot{Y}_i = 0$$

$$\Downarrow$$

Veličina $I(\vec{q},\dot{\vec{q}},t) = \sum_{k=1}^{s} \frac{\partial L}{\partial \dot{q}_k} Y_k$ je I. P.

Pozn. $L'(\vec{q},\dot{\vec{q}},t,\varepsilon) = L'(\vec{q},\dot{\vec{q}},t,0) + \frac{\partial L'}{\partial \varepsilon}\Big|_{\varepsilon=0}\cdot\varepsilon + \mathcal{O}(\varepsilon^2)$ $\qquad$ $0 = \frac{\partial L'}{\partial \varepsilon}\Big|_{\varepsilon=0}\cdot\varepsilon = \mathit{lin.\,část}\left[L'(\vec{q},\dot{\vec{q}},t,\varepsilon) - L(\vec{q},\dot{\vec{q}},t)\right]$

Infinitesimální verze teorému Noetherové:

Transfromace (Taylorův rozvoj do 1. řádu v ε)

$$q'_j = q'_j(\vec{q},0) + \frac{\partial q'_j}{\partial \varepsilon}\Big|_{\varepsilon=0}\cdot\varepsilon + \mathcal{O}(\varepsilon^2) = q_j + Y_j\cdot\varepsilon \qquad \underbrace{Y_j = \frac{\partial q'_j}{\partial \varepsilon}\Big|_{\varepsilon=0} = Y_j(\vec{q})}_{\text{vektorové pole tzv. generátor transformace}}$$

$$\dot{q}'_j = \frac{\hat{d} q'_j}{dt} = \dot{q}_j + \dot{Y}_j\cdot\varepsilon \quad \text{kde } \dot{Y}_j = \frac{\partial Y_j}{\partial q_i}\dot{q}_i$$

Invariance L do 1. řádu v ε $\quad \forall\vec{q}\ \forall\dot{\vec{q}}\ \forall t$

$$\frac{\partial L'}{\partial \varepsilon}\Big|_{\varepsilon=0} = \frac{\partial L}{\partial q_i}Y_i + \frac{\partial L}{\partial \dot{q}_i}\dot{Y}_i = 0$$

$$\Downarrow$$

Veličina $I(\vec{q},\dot{\vec{q}},t) = \sum_{k=1}^{\Delta} \frac{\partial L}{\partial \dot{q}_k}Y_k$ je I. P.

Př. rotace kolem osy x_3

$$x'_1 = x_1\cos\varepsilon - x_2\sin\varepsilon$$
$$x'_2 = x_1\sin\varepsilon + x_2\cos\varepsilon$$
$$x'_3 = x_3$$

Pozn. $L'(\vec{q},\dot{\vec{q}},t,\varepsilon) = L'(\vec{q},\dot{\vec{q}},t,0) + \left.\frac{\partial L'}{\partial \varepsilon}\right|_{\varepsilon=0} \cdot \varepsilon + \mathcal{O}(\varepsilon^2)$ $\qquad 0 = \left.\frac{\partial L'}{\partial \varepsilon}\right|_{\varepsilon=0} \cdot \varepsilon = \text{lin. část}\left[L'(\vec{q},\dot{\vec{q}},t,\varepsilon) - L(\vec{q},\dot{\vec{q}},t)\right]$

Infinitesimální verze teorému Noetherové:

Transfromace (Taylorův rozvoj do 1. řádu v ε)

$$q'_j = q'_j(\vec{q},0) + \left.\frac{\partial q'_j}{\partial \varepsilon}\right|_{\varepsilon=0} \cdot \varepsilon + \mathcal{O}(\varepsilon^2) = q_j + Y_j \cdot \varepsilon \qquad \underbrace{Y_j = \left.\frac{\partial q'_j}{\partial \varepsilon}\right|_{\varepsilon=0} = Y_j(\vec{q})}_{\text{vektorové pole tzv. generátor transformace}}$$

$$\dot{q}'_j = \frac{\hat{d} q'_j}{dt} = \dot{q}_j + \dot{Y}_j \cdot \varepsilon \quad \text{kde} \quad \dot{Y}_j = \frac{\partial Y_j}{\partial q_i} \dot{q}_i$$

Invariance L do 1. řádu v ε $\quad \forall \vec{q}\ \forall \dot{\vec{q}}\ \forall t$

$$\left.\frac{\partial L'}{\partial \varepsilon}\right|_{\varepsilon=0} = \frac{\partial L}{\partial q_i} Y_i + \frac{\partial L}{\partial \dot{q}_i} \dot{Y}_i = 0$$

$$\Downarrow$$

Veličina $I(\vec{q},\dot{\vec{q}},t) = \sum_{k=1}^{s} \frac{\partial L}{\partial \dot{q}_k} Y_k$ je I. P.

Př. rotace kolem osy x_3

$$x'_1 = x_1 \cos\varepsilon - x_2 \sin\varepsilon \qquad Y_1 = \left(\frac{\partial x'_1}{\partial \varepsilon}\right)_{\varepsilon=0} = (-x_1 \sin\varepsilon - x_2 \cos\varepsilon)_{\varepsilon=0} = -x_2$$

$$x'_2 = x_1 \sin\varepsilon + x_2 \cos\varepsilon \qquad Y_2 = \left(\frac{\partial x'_2}{\partial \varepsilon}\right)_{\varepsilon=0} = (x_1 \cos\varepsilon - x_2 \sin\varepsilon)_{\varepsilon=0} = x_1$$

$$x'_3 = x_3 \qquad Y_3 = \left(\frac{\partial x'_3}{\partial \varepsilon}\right)_{\varepsilon=0} = 0$$

Pozn. $L'(\vec{q},\dot{\vec{q}},t,\varepsilon) = L'(\vec{q},\dot{\vec{q}},t,0) + \left.\frac{\partial L'}{\partial \varepsilon}\right|_{\varepsilon=0} \cdot \varepsilon + \mathcal{O}(\varepsilon^2)$ $\qquad$ $0 = \left.\frac{\partial L'}{\partial \varepsilon}\right|_{\varepsilon=0} \cdot \varepsilon = \text{lin. část}\left[L'(\vec{q},\dot{\vec{q}},t,\varepsilon) - L(\vec{q},\dot{\vec{q}},t)\right]$

<u>Infinitesimální verze teorému Noetherové:</u>

Transfromace (Taylorův rozvoj do 1. řádu v ε)

$$q'_j = q'_j(\vec{q},0) + \left.\frac{\partial q'_j}{\partial \varepsilon}\right|_{\varepsilon=0} \cdot \varepsilon + \mathcal{O}(\varepsilon^2) = q_j + Y_j \cdot \varepsilon \qquad \underbrace{Y_j = \left.\frac{\partial q'_j}{\partial \varepsilon}\right|_{\varepsilon=0} = Y_j(\vec{q})}_{\text{vektorové pole tzv. generátor transformace}}$$

$$\dot{q}'_j = \frac{\hat{d} q'_j}{dt} = \dot{q}_j + \dot{Y}_j \cdot \varepsilon \quad \text{kde } \dot{Y}_j = \frac{\partial Y_j}{\partial q_i}\dot{q}_i$$

Invariance L do 1. řádu v ε $\quad \forall \vec{q}\ \forall \dot{\vec{q}}\ \forall t$

$$\left.\frac{\partial L'}{\partial \varepsilon}\right|_{\varepsilon=0} = \frac{\partial L}{\partial q_i} Y_i + \frac{\partial L}{\partial \dot{q}_i} \dot{Y}_i = 0$$

$$\Downarrow$$

Veličina $I(\vec{q},\dot{\vec{q}},t) = \sum_{k=1}^{\triangle} \frac{\partial L}{\partial \dot{q}_k} Y_k$ je I. P.

Př. rotace kolem osy x_3

$$x'_1 = x_1 \cos\varepsilon - x_2 \sin\varepsilon \qquad Y_1 = \left(\frac{\partial x'_1}{\partial \varepsilon}\right)_{\varepsilon=0} = (-x_1 \sin\varepsilon - x_2\cos\varepsilon)_{\varepsilon=0} = -x_2$$

$$x'_2 = x_1 \sin\varepsilon + x_2 \cos\varepsilon \qquad Y_2 = \left(\frac{\partial x'_2}{\partial \varepsilon}\right)_{\varepsilon=0} = (x_1 \cos\varepsilon - x_2\sin\varepsilon)_{\varepsilon=0} = x_1$$

$$x'_3 = x_3 \qquad Y_3 = \left(\frac{\partial x'_3}{\partial \varepsilon}\right)_{\varepsilon=0} = 0$$

Pokud

$$L(\vec{x}',\dot{\vec{x}}',t) = L(\vec{x},\dot{\vec{x}},t) \Rightarrow I = \sum_{i=1}^{\triangle} \frac{\partial L}{\partial \dot{x}_i} Y_i = p_i Y_i = p_1(-x_2) + p_2 x_1 + 0 = L_3$$

Pozn. $L'(\vec{q},\dot{\vec{q}},t,\varepsilon) = L'(\vec{q},\dot{\vec{q}},t,0) + \frac{\partial L'}{\partial \varepsilon}\Big|_{\varepsilon=0}\cdot\varepsilon + \mathcal{O}(\varepsilon^2)$ $\qquad 0 = \frac{\partial L'}{\partial \varepsilon}\Big|_{\varepsilon=0}\cdot\varepsilon = \text{lin. část}\Big[L'(\vec{q},\dot{\vec{q}},t,\varepsilon) - L(\vec{q},\dot{\vec{q}},t)\Big]$

Infinitesimálni verze teorému Noetherové:

Transfromace (Taylorův rozvoj do 1. řádu v ε)

$$q'_j = q'_j(\vec{q},0) + \frac{\partial q'_j}{\partial \varepsilon}\Big|_{\varepsilon=0}\cdot\varepsilon + \mathcal{O}(\varepsilon^2) = q_j + Y_j\cdot\varepsilon \qquad \underbrace{Y_j = \frac{\partial q'_j}{\partial \varepsilon}\Big|_{\varepsilon=0} = Y_j(\vec{q})}_{\text{vektorové pole tzv. generátor transformace}}$$

$$\dot{q}'_j = \frac{\hat{d} q'_j}{dt} = \dot{q}_j + \dot{Y}_j\cdot\varepsilon \quad \text{kde } \dot{Y}_j = \frac{\partial Y_j}{\partial q_i}\dot{q}_i$$

Invariance L do 1. řádu v ε $\quad \forall \vec{q}\ \forall \dot{\vec{q}}\ \forall t$

$$\frac{\partial L'}{\partial \varepsilon}\Big|_{\varepsilon=0} = \frac{\partial L}{\partial q_i}Y_i + \frac{\partial L}{\partial \dot{q}_i}\dot{Y}_i = 0$$

$$\Downarrow$$

Veličina $I(\vec{q},\dot{\vec{q}},t) = \sum_{k=1}^{s}\frac{\partial L}{\partial \dot{q}_k}Y_k$ je I. P.

Př. rotace kolem osy x_3

$$x'_1 = x_1\cos\varepsilon - x_2\sin\varepsilon \qquad Y_1 = \left(\frac{\partial x'_1}{\partial \varepsilon}\right)_{\varepsilon=0} = (-x_1\sin\varepsilon - x_2\cos\varepsilon)_{\varepsilon=0} = -x_2$$

$$x'_2 = x_1\sin\varepsilon + x_2\cos\varepsilon \qquad Y_2 = \left(\frac{\partial x'_2}{\partial \varepsilon}\right)_{\varepsilon=0} = (x_1\cos\varepsilon - x_2\sin\varepsilon)_{\varepsilon=0} = x_1$$

$$x'_3 = x_3 \qquad Y_3 = \left(\frac{\partial x'_3}{\partial \varepsilon}\right)_{\varepsilon=0} = 0$$

Infinitesimálně

$$x'_1 = x_1 - x_2\varepsilon$$
$$x'_2 = x_2 + x_1\varepsilon$$
$$x'_3 = x_3$$

Pokud

$$L(\vec{x}',\dot{\vec{x}}',t) = L(\vec{x},\dot{\vec{x}},t) \Rightarrow I = \sum_{i=1}^{s}\frac{\partial L}{\partial \dot{x}_i}Y_i = p_i Y_i = p_1(-x_2) + p_2 x_1 + 0 = L_3$$

(pro infinitesimalní do 1. řádu v ε)

Pozn. $L'(\vec{q},\dot{\vec{q}},t,\varepsilon) = L'(\vec{q},\dot{\vec{q}},t,0) + \left.\frac{\partial L'}{\partial \varepsilon}\right|_{\varepsilon=0}\cdot\varepsilon + \mathcal{O}(\varepsilon^2)$ $\quad 0 = \left.\frac{\partial L'}{\partial \varepsilon}\right|_{\varepsilon=0}\cdot\varepsilon = \text{lin. část}\left[L'(\vec{q},\dot{\vec{q}},t,\varepsilon) - L(\vec{q},\dot{\vec{q}},t)\right]$

Infinitesimální verze teorému Noetherové:

Transfromace (Taylorův rozvoj do 1. řádu v ε)

$$q'_j = q'_j(\vec{q},0) + \left.\frac{\partial q'_j}{\partial \varepsilon}\right|_{\varepsilon=0}\cdot\varepsilon + \mathcal{O}(\varepsilon^2) = q_j + Y_j\cdot\varepsilon \qquad \underbrace{Y_j = \left.\frac{\partial q'_j}{\partial \varepsilon}\right|_{\varepsilon=0} = Y_j(\vec{q})}_{\text{vektorové pole tzv. generátor transformace}}$$

$$\dot{q}'_j = \frac{\hat{d} q'_j}{dt} = \dot{q}_j + \dot{Y}_j\cdot\varepsilon \quad \text{kde } \dot{Y}_j = \frac{\partial Y_j}{\partial q_i}\dot{q}_i$$

Invariance L do 1. řádu v ε $\quad \forall\vec{q}\ \forall\dot{\vec{q}}\ \forall t$

$$\left.\frac{\partial L'}{\partial \varepsilon}\right|_{\varepsilon=0} = \frac{\partial L}{\partial q_i}Y_i + \frac{\partial L}{\partial \dot{q}_i}\dot{Y}_i = 0$$

$$\Downarrow$$

Veličina $I(\vec{q},\dot{\vec{q}},t) = \sum_{k=1}^{\Delta}\frac{\partial L}{\partial \dot{q}_k}Y_k$ je I. P.

Př. rotace kolem osy x_3

$$x'_1 = x_1\cos\varepsilon - x_2\sin\varepsilon \qquad Y_1 = \left(\frac{\partial x'_1}{\partial\varepsilon}\right)_{\varepsilon=0} = (-x_1\sin\varepsilon - x_2\cos\varepsilon)_{\varepsilon=0} = -x_2$$

$$x'_2 = x_1\sin\varepsilon + x_2\cos\varepsilon \qquad Y_2 = \left(\frac{\partial x'_2}{\partial\varepsilon}\right)_{\varepsilon=0} = (x_1\cos\varepsilon - x_2\sin\varepsilon)_{\varepsilon=0} = x_1$$

$$x'_3 = x_3 \qquad Y_3 = \left(\frac{\partial x'_3}{\partial\varepsilon}\right)_{\varepsilon=0} = 0$$

Infinitesimálně

$$x'_1 = x_1 - x_2\varepsilon$$
$$x'_2 = x_2 + x_1\varepsilon$$
$$x'_3 = x_3$$

$$\vec{x}' = \begin{pmatrix}1 & -\varepsilon & 0\\ \varepsilon & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\vec{x} = \mathbb{1}\vec{x} + \varepsilon\overbrace{\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}}^{\mathbb{A}}\vec{x}$$

Pokud

$$L(\vec{x}',\dot{\vec{x}}',t) = L(\vec{x},\dot{\vec{x}},t) \Rightarrow I = \sum_{i=1}^{\Delta}\frac{\partial L}{\partial \dot{x}_i}Y_i = p_i Y_i = p_1(-x_2) + p_2 x_1 + 0 = L_3$$

(pro infinitesimalní do 1. řádu v ε)

Pozn. $L'(\vec{q},\dot{\vec{q}},t,\varepsilon) = L'(\vec{q},\dot{\vec{q}},t,0) + \frac{\partial L'}{\partial \varepsilon}\Big|_{\varepsilon=0} \cdot \varepsilon + \mathcal{O}(\varepsilon^2)$ $\qquad 0 = \frac{\partial L'}{\partial \varepsilon}\Big|_{\varepsilon=0} \cdot \varepsilon = \text{lin. část}\left[L'(\vec{q},\dot{\vec{q}},t,\varepsilon) - L(\vec{q},\dot{\vec{q}},t)\right]$

Infinitesimální verze teorému Noetherové:

Transfromace (Taylorův rozvoj do 1. řádu v ε)

$$q'_j = q'_j(\vec{q},0) + \frac{\partial q'_j}{\partial \varepsilon}\Big|_{\varepsilon=0} \cdot \varepsilon + \mathcal{O}(\varepsilon^2) = q_j + Y_j \cdot \varepsilon \qquad \underbrace{Y_j = \frac{\partial q'_j}{\partial \varepsilon}\Big|_{\varepsilon=0} = Y_j(\vec{q})}_{\text{vektorové pole tzv. generátor transformace}}$$

$$\dot{q}'_j = \frac{\hat{d} q'_j}{dt} = \dot{q}_j + \dot{Y}_j \cdot \varepsilon \quad \text{kde } \dot{Y}_j = \frac{\partial Y_j}{\partial q_i} \dot{q}_i$$

Invariance L do 1. řádu v ε $\quad \forall \vec{q}\ \forall \dot{\vec{q}}\ \forall t$

$$\frac{\partial L'}{\partial \varepsilon}\Big|_{\varepsilon=0} = \frac{\partial L}{\partial q_i} Y_i + \frac{\partial L}{\partial \dot{q}_i} \dot{Y}_i = 0$$

$$\Downarrow$$

Veličina $I(\vec{q},\dot{\vec{q}},t) = \sum_{k=1}^{s} \frac{\partial L}{\partial \dot{q}_k} Y_k$ je I. P.

Př. rotace kolem osy x_3

$$x'_1 = x_1 \cos\varepsilon - x_2 \sin\varepsilon$$
$$x'_2 = x_1 \sin\varepsilon + x_2 \cos\varepsilon$$
$$x'_3 = x_3$$

$$Y_1 = \left(\frac{\partial x'_1}{\partial \varepsilon}\right)_{\varepsilon=0} = (-x_1 \sin\varepsilon - x_2 \cos\varepsilon)_{\varepsilon=0} = -x_2$$
$$Y_2 = \left(\frac{\partial x'_2}{\partial \varepsilon}\right)_{\varepsilon=0} = (x_1 \cos\varepsilon - x_2 \sin\varepsilon)_{\varepsilon=0} = x_1$$
$$Y_3 = \left(\frac{\partial x'_3}{\partial \varepsilon}\right)_{\varepsilon=0} = 0$$

Infinitesimálně

$$x'_1 = x_1 - x_2 \varepsilon$$
$$x'_2 = x_2 + x_1 \varepsilon$$
$$x'_3 = x_3$$

$$\vec{x}' = \begin{pmatrix} 1 & -\varepsilon & 0 \\ \varepsilon & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \vec{x} = \mathbb{1}\vec{x} + \varepsilon \overbrace{\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}^{\mathbb{A}} \vec{x}$$

Pokud

$$L(\vec{x}',\dot{\vec{x}}',t) = L(\vec{x},\dot{\vec{x}},t) \Rightarrow I = \sum_{i=1}^{s} \frac{\partial L}{\partial \dot{x}_i} Y_i = p_i Y_i = p_1(-x_2) + p_2 x_1 + 0 = L_3$$

(pro infinitesimalní do 1. řádu v ε)

Pozn. původní transformace $\vec{x}' = \exp(\varepsilon\mathbb{A})\vec{x}$

$$\exp(\varepsilon\mathbb{A}) = \sum_{k=0}^{+\infty} \frac{\varepsilon^k \mathbb{A}^k}{k!} = \begin{pmatrix} \cos\varepsilon & -\sin\varepsilon & 0 \\ \sin\varepsilon & \cos\varepsilon & 0 \\ 0 & 0 & 1 \end{pmatrix}$$